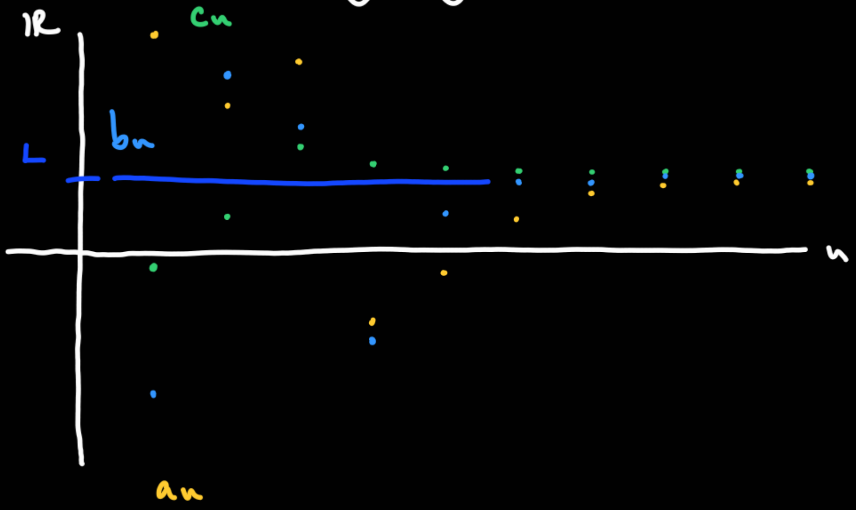
11.1. Sequences (continued)

Squeeze Theorem: Let $\{a_n\}, \{b_n\}, \{c_n\}$ with $a_n \leq b_n \leq c_n$ from some point onwards (i.e. for n big enough) and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = L$.



Example: Compute $\lim_{n\to\infty} \frac{R^n}{n!}$ for R any real number.

Trying to compute this limit using: $a_n = f(n) = \frac{R^n}{n!}$, $f(x) = \frac{R^x}{x!}$, we find that $f(x)$ is not a continuous function. We do not know what $x!$ is, it is not a function.

If $R=0$ then $\frac{R^n}{n!} = 0$ so $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.

Consider $R>0$. We are computing $\lim_{n\to\infty} \frac{R^n}{n!}$.

Quick analysis: $\frac{R^n}{n!} = \frac{\overbrace{R\cdot R\cdot R \cdots R}^{n}}{1\cdot 2\cdot 3 \cdots (n-1)\cdot n}$. Intuitively dividing by a large number n that goes to infinity makes that fraction go to zero.

Since R is a positive real number, there is a natural number M such that $M \le R < M+1$.

Now for n much bigger than M:

$$0 < \frac{R^n}{n!} = \frac{R\cdot R\cdot R \cdots R\cdot R \cdots R\cdot R}{1\cdot 2\cdot 3 \cdots M\cdot(M+1)\cdots(n-1)\cdot n} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdot\frac{R}{3}\cdots\frac{R}{M}}_{c_1}\cdot \overbrace{\underbrace{\underbrace{\frac{R}{M+1}}_{<1}\cdots\underbrace{\frac{R}{n-1}}_{<1}}_{<1}}^{A}\cdot\underbrace{\frac{R}{n}}_{\text{keep this term.}} =$$

$$\begin{matrix} M \le R & & 1 \le \frac{R}{M} \\ & \rightsquigarrow & \\ R < M+1 & & \frac{R}{M+1} < 1 \end{matrix}$$

$$= c_1 \cdot A \cdot \frac{R}{n} \underset{\uparrow}{<} c_1 \cdot \frac{R}{n}.$$

if $A<1$ and B is a number, then $A\cdot B < B$.

Since $\lim_{n\to\infty} c_1 \cdot \frac{R}{n} = 0 = \lim_{n\to\infty} 0$, take $a_n = 0$, take $c_n = c_1\cdot\frac{R}{n}$, take $b_n = \frac{R^n}{n!}$. By the Squeeze Theorem:

$$\lim_{n\to\infty} \frac{R^n}{n!} = \lim_{n\to\infty} b_n = 0.$$

<u>Sketch:</u> for $R<0$ we take: $-\frac{|R|^n}{n!} < \frac{R^n}{n!} \le \frac{|R|^n}{n!}$.

$$\underset{\substack{|R|>0 \;\downarrow\\ -0=0}}{-\frac{|R|^n}{n!}} \qquad \underset{\substack{\downarrow\\ 0}}{\frac{R^n}{n!}} \qquad \underset{\substack{\downarrow\; |R|>0\\ 0}}{\frac{|R|^n}{n!}}$$

This is the same idea used to compute $\lim_{n\to\infty} c\cdot r^n = 0$ for $-1<r<0$.

<u>Think about solving with squeeze theorem:</u>

(a) $\lim_{n\to\infty} \frac{\sin(n)}{n^2}$. (b) $\lim_{n\to\infty} (-1)^{n+1}\cdot \frac{e^{-3n}}{n^2}$.

We can treat converging limits as numbers. Converging limits behave like numbers.

Limit laws for sequences: Let $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$.

(i) $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$.

(ii) $\lim_{n\to\infty} (a_n \cdot b_n) = L \cdot M$.

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

(iv) $\lim_{n\to\infty} c_1 \cdot a_n = c_1 \cdot L$ for c_1 constant.

Example: Compute:

$$\lim_{n\to\infty} \frac{2n^2-3}{8n+5n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\cdot\left(2-\frac{3}{n^2}\right)}{\cancel{n^2}\cdot\left(5+\frac{8}{n}\right)} = \frac{\lim_{n\to\infty}\left(2-\frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(5+\frac{8}{n}\right)} = \frac{2}{5}.$$

Example: Compute:

$$\lim_{n\to\infty}\left(\sqrt[3]{\frac{2n+3}{n}} - \frac{1}{n}\right) = \lim_{n\to\infty}\sqrt[3]{\frac{2n+3}{n}} - \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{0} =$$

$$= \sqrt[3]{\lim_{n\to\infty}\frac{2n+3}{n}} = \sqrt[3]{\lim_{n\to\infty}\frac{\cancel{n}\cdot\left(2+\frac{3}{n}\right)}{\cancel{n}}} = \sqrt[3]{\lim_{n\to\infty}\left(2+\frac{3}{n}\right)} = \sqrt[3]{2}.$$

$$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin(x)}{\cos(x)} =$$

The hypothesis of LHR are satisfied, we use it.