11.2. Summing an infinite series.

An infinite series is a sum $\sum_{n=0}^{\infty} a_n$ where $\{a_n\}$ is a sequence.

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots$$

Example: $a_n = n$, then $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots = 0 + 1 + 2 + 3 + \cdots$ is not finite.

Example: $a_n = \frac{(-1)^n}{2\cdot n+1}$, then $\sum_{n=0}^{\infty} a_n = \underbrace{\frac{1}{2\cdot 0+1}}_{a_0} - \underbrace{\frac{1}{2\cdot 1+1}}_{a_1} + \underbrace{\frac{1}{2\cdot 2+1}}_{a_2} - \underbrace{\frac{1}{2\cdot 3+1}}_{a_3} + \cdots =$

$$= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}.$$

arctan is involved.

The sum S of an infinite series is:

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N\to\infty}\Big(\underbrace{\sum_{n=0}^{N} a_n}_{S_N}\Big) = \lim_{N\to\infty} S_N$$

definition

S_N partial sums

$$\int_0^{\infty} f(x)\,dx = \lim_{R\to\infty}\left(\int_0^{R} f(x)\,dx\right).$$

When the limit converges we say that the infinite series converges to S a finite real number. If the limit does not converge, we say that the infinite series diverges.

To sum an infinite series $\sum_{n=1}^{\infty} a_n$, we begin with a sequence $\{a_n\}$. We then compute the partial sums $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N$. All together, these form the sequence of partial sums $\{S_N\}$. The sum of the infinite series is the limit of the sequence of partial sums: $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$.

Example: Compute $\sum_{n=0}^{\infty} a_n$ for $a_n = n$.

This is: $\sum_{n=0}^{\infty} n = 0+1+2+3+\cdots$.

We know: $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\left(\sum_{n=0}^{N} a_n\right)}_{S_N} = \lim_{N\to\infty} S_N$.

$S_0 = a_0 = 0$ $\qquad N=0 \qquad \frac{0\cdot(0+1)}{2} = 0$

$S_1 = a_0 + a_1 = 0+1 = 1$ $\qquad N=1 \qquad \frac{1\cdot(1+1)}{2} = 1$

$S_2 = a_0 + a_1 + a_2 = 0+1+2 = 3$ $\qquad N=2 \qquad \frac{2\cdot(2+1)}{2} = 3$

$S_3 = a_0 + a_1 + a_2 + a_3 = 0+1+2+3 = 6$ $\qquad N=3 \qquad \frac{3\cdot(3+1)}{2} = 6.$

$\vdots$

Gauss

$S_N = a_0 + a_1 + \cdots + a_{N-1} + a_N = \underbrace{0+1+2+\cdots+N-1+N}_{\text{the sum of the first } N \text{ numbers.}} = \frac{N\cdot(N+1)}{2}$

Now:

$$\sum_{n=0}^{\infty} n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N\cdot(N+1)}{2} = \infty,$$ the infinite series diverges.

Example: Sum the sequence $a_n = (-1)^n$ from $n=0$ to infinity.

Namely, compute $\sum_{n=0}^{\infty} (-1)^n$. We know: $\sum_{n=0}^{\infty} (-1)^n = \lim_{N\to\infty} \underbrace{\left(\sum_{n=0}^{N} (-1)^n\right)}_{S_N}$.

We find the general term for S_N:

$$S_0 = (-1)^0 = 1$$

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\left(\int_0^R f(x)dx\right)$$

$$S_1 = (-1)^0 + (-1)^1 = 1 - 1 = 0$$

$$S_2 = (-1)^0 + (-1)^1 + (-1)^2 = 1 - 1 + 1 = 1$$

$\vdots$

$$S_N = \begin{cases} 1 & N \text{ even.} \\ 0 & N \text{ odd.} \end{cases}$$

$$\{a_n\} \rightsquigarrow \underbrace{a_0}_{S_0}, a_1, a_2, a_3, a_4, \dots$$

$$\{S_N\} \rightsquigarrow \underbrace{a_0, a_1}_{S_1}, \quad \underbrace{a_0, a_1, a_2}_{S_2} \quad \dots$$

Now: $\displaystyle\sum_{n=0}^{\infty} (-1)^n = \lim_{N\to\infty} S_N$ does not exist. The sum diverges.

<u>Example:</u> <u>Telescopic series</u>

Compute the sum: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\cdot(n+2)}$.

We have $a_n = \dfrac{2}{n\cdot(n+2)}$.

We may be tempted to compute $S_0, S_1, S_2, \dots$ and find a pattern,

but this will be very hard or impossible.

check!

We can use partial fraction decomposition to get: $\dfrac{2}{n\cdot(n+2)} = \dfrac{1}{n} - \dfrac{1}{n+2}$.

$$\frac{A}{n} + \frac{B}{n+2}$$

$A = 1$
$B = -1$

We can now find the general term for S_N:

$$S_N = a_1 + a_2 + a_3 + a_4 + \cdots + a_{N-2} + a_{N-1} + a_N =$$

$$= \left(\frac{1}{1} - \frac{1}{1+2}\right) + \left(\frac{1}{2} - \frac{1}{2+2}\right) + \left(\frac{1}{3} - \frac{1}{3+2}\right) + \left(\frac{1}{4} - \frac{1}{4+2}\right) + \cdots +$$

$$+ \left(\frac{1}{N-2} - \frac{1}{N-2+2}\right) + \left(\frac{1}{N-1} - \frac{1}{N-1+2}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right) =$$

$$= 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \cdots +$$

$$\frac{1}{5} \qquad \frac{1}{6}$$

$$+ \cancel{\frac{1}{N-2}} - \cancel{\frac{1}{N}} + \cancel{\frac{1}{N-1}} - \frac{1}{N+1} + \cancel{\frac{1}{N}} - \frac{1}{N+2} = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$-\frac{1}{N-1}$$

Now:

$$\sum_{n=1}^{\infty} \frac{2}{n\cdot(n+2)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = 1 + \frac{1}{2} = \frac{3}{2}.$$