11.1.28.: $a_n = e^{\frac{4n}{3n+9}}$. Determine the limit.

Theorem 1: If $a_n = f(n)$ with $f(x)$ continuous and $\lim_{x\to\infty} f(x) = L$ finite,

then: $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = L$.

Theorem 4: If $\lim_{n\to\infty} a_n = L$ and $f(x)$ continuous then:

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

We can bring limits inside continuous functions.

We have $a_n = e^{\frac{4n}{3n+9}} = f(n)$ giving $f(x) = e^{\frac{4x}{3x+9}}$ a continuous function.

Also: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\frac{4x}{3x+9}} = e^{\lim_{x\to\infty}\frac{4x}{3x+9}} = e^{\frac{4}{3}}$. Then:

$$\lim_{n\to\infty} a_n \underset{\text{Theorem 1.}}{=} \lim_{x\to\infty} f(x) = e^{\frac{4}{3}}.$$

To apply Theorem 4, we need to want to compute $\lim_{n\to\infty} f(a_n)$ and we need to know $\lim_{n\to\infty} a_n = L$ finite. We are told to compute the limit of $e^{\frac{4n}{3n+9}}$.

We will then have $f(a_n) = e^{\frac{4n}{3n+9}}$. We now write $e^{\frac{4n}{3n+9}}$ as a continuous function $f(x)$ evaluated at some sequence a_n. We chose $f(x) = e^x$. We choose $a_n = \frac{4n}{3n+9}$. Now indeed: $f(a_n) = e^{a_n} = e^{\frac{4n}{3n+9}}$ and

$$\lim_{n\to\infty} \frac{4n}{3n+9} \overset{\text{Theorem 1}}{=} \lim_{x\to\infty} \frac{4x}{3x+9} = \frac{4}{3}.$$ So by Theorem 4:

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f\left(\lim_{n\to\infty} \frac{4n}{3n+9}\right) = f\left(\frac{4}{3}\right) = e^{\frac{4}{3}}.$$

8.6.40.: Integrate: $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\cdot dx.$

This is an improper integral because of the ∞. Then:

$$\int_1^\infty \frac{\ln(x)}{x^2}\cdot dx \overset{\text{definition}}{=} \lim_{R\to\infty}\left(\int_1^R \frac{\ln(x)}{x^2}\cdot dx\right) = \lim_{R\to\infty}\left(\frac{-\ln(x)-1}{x}\Bigg|_1^R\right) =$$

$$\int \frac{\ln(x)}{x^2}\cdot dx \overset{\text{IBP}}{=} \ln(x)\cdot\frac{-1}{x} - \int \frac{-1}{x}\cdot\frac{1}{x}\cdot dx = -\frac{\ln(x)}{x} + \int \frac{dx}{x^2} = \frac{-\ln(x)}{x} - \frac{1}{x}$$

$u = \ln(x)$ $\quad du = \frac{1}{x}\cdot dx$ $\qquad \int u\cdot dv = u\cdot v - \int v\cdot du$

$dv = \frac{1}{x^2}\cdot dx$ $\quad v = \frac{-1}{x}$

$$= \lim_{R\to\infty}\left(\frac{-\ln(R)-1}{R} - \underbrace{\frac{-\ln(1)-1}{1}}_{+1}\right) = 1 + \underbrace{\lim_{R\to\infty}\left(\frac{-\ln(R)-1}{R}\right)}_{\substack{0\\ \text{LHR}}} = 1.$$

Midterm 1.4: Find: $\displaystyle\lim_{x\to\frac{\pi}{2}}\left(\sec(x) - \tan(x)\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) =$

" $\sec\left(\frac{\pi}{2}\right) = \frac{1}{0} = \infty$ " " $\tan\left(\frac{\pi}{2}\right) = \frac{1}{0} = \infty$ " We have $\infty - \infty$.

$$= \lim_{x\to\frac{\pi}{2}}\left(\frac{1-\sin(x)}{\cos(x)}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{0-\cos(x)}{-\sin(x)}\right) = \lim_{x\to\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)} = 0.$$

$1-\sin\left(\frac{\pi}{2}\right) = 1-1 = 0$ $\quad \frac{0}{0}$ $\quad$ LHR

$\cos\left(\frac{\pi}{2}\right) = 0$

9.4.42.: Find n for which $|\ln(1.3) - T_n(1.3)| \le 10^{-4}$, $a = 1$.

The error bound says: $\displaystyle |\ln(x) - T_n(x)| \le K\cdot\frac{|x-a|^{n+1}}{(n+1)!} = K\cdot\frac{|1.3-1|^{n+1}}{(n+1)!} = K\cdot\frac{\left(\frac{3}{10}\right)^{n+1}}{(n+1)!}.$

$x = 1.3$

$a = 1$

Where k is the max of $|f^{(n+1)}(x)|$ for x between x and a.

$f(x) = \ln(x)$ $\qquad$ $n=0$

$f'(x) = \frac{1}{x}$ $\qquad$ $f^{(n)}(x) = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}$

$f''(x) = \frac{-1}{x^2}$ $\qquad$ $|f^{(n+1)}(x)| = \left| \frac{(-1)^{(n+1)+1} \cdot (n+1-1)!}{x^{n+1}} \right| = \frac{n!}{x^{n+1}}$

$f'''(x) = \frac{1\cdot 2}{x^3}$ $\qquad$ $k = \max \frac{n!}{x^{n+1}}$ between 1 and 1.3. Since

$f^{IV}(x) = \frac{-1\cdot 2\cdot 3}{x^4}$ $\qquad$ $\frac{n!}{x^{n+1}}$ is decreasing, it has max at $x=1$.

$$k = \frac{n!}{1^{n+1}} = n!$$

Putting this back in the error bound:

$$|\ln(x) - T_n(x)| \leq k \cdot \frac{\left(\frac{3}{10}\right)^{n+1}}{(n+1)!} = \frac{n! \cdot 3^{n+1}}{(n+1)! \cdot 10^{n+1}} = \frac{3^{n+1}}{(n+1) \cdot 10^{n+1}} < 10^{-4} = \frac{1}{10000}$$

$(n+1)! = (n!) \cdot (n+1)$

plug in $n = 1, 2, 3, \ldots$ and keep the smallest.

9.4.54.:

$\boxed{9.4.53}$

$f(x)$ poly. of degree n. What is $T_n(x)$?

$f(x) = 2 \cdot x^3 + 5 \cdot x^2 + 1$ $\qquad$ $f(x)$ poly of degree n.

$f'(x) = 6 \cdot x^2 + 10x$ $\qquad$ All derivatives above

$f''(x) = 12 \cdot x + 10$ $\qquad$ $f^{(n)}(x)$ will be zero.

$f'''(x) = 12$, $\quad f^{IV}(x) = 0$ and all others also.

$T_n(x) = f(x)$ for $f(x)$ poly. of degree n around <u>all</u> a.

<u>Try computing T_1, T_2, T_3, T_4, T_5 of $f(x) = x^4 - 1$ around $a = 1$.</u> and simplify!

<u>8.5.56.:</u> Compute $\int x \cdot \sec^2(x) dx$.

<u>8.6.56.:</u> Show that $\int_2^\infty \frac{dx}{x^3 - 4}$ converges, by comparing with $\int_2^\infty 2 \cdot x^{-3} \cdot dx$.

Note: $\int_2^\infty 2 \cdot x^{-3} \cdot dx = \int_2^\infty \frac{2}{x^3} \cdot dx = 2 \cdot \int_2^\infty \frac{dx}{x^3}$ converging p-integral. $\left(\int \frac{1}{x^p} \cdot dx \cdot \right)$

To apply the Comparison Theorem we need: $\int_2^\infty \frac{dx}{x^3 - 4} < \int_2^\infty \frac{2}{x^3} \cdot dx$, it suffices $\frac{1}{x^3 - 4} < \frac{2}{x^3}$ from some real number M onwards.

Is it true $\frac{1}{x^3 - 4} < \frac{2}{x^3}$?

$$\frac{1}{x^3 - 4} \overset{?}{<} \frac{2}{x^3} \rightsquigarrow x^3 \overset{?}{<} 2 \cdot (x^3 - 4) \rightsquigarrow x^3 \overset{?}{<} 2x^3 - 8$$

$$\rightsquigarrow 0 \overset{?}{<} 2x^3 - x^3 - 8 \rightsquigarrow 0 \overset{?}{<} x^3 - 8 \rightsquigarrow 8 \overset{?}{<} x^3$$

In our interval of integration $(2, \infty)$ we indeed have $8 < x^3$. So it is true that for x in $(2, \infty)$ we have $\frac{1}{x^3 - 4} < \frac{2}{x^3}$. The Comparison Theorem applies.

<u>9.4.49.:</u> $f(x) = e^x \cdot \sin(x)$ $a = 0$ $f(0) = 0$, $f'(x) = e^x \cdot \sin(x) + e^x \cdot \cos(x)$ $f'(0) = 1$,

$f''(x) = \cancel{e^x \cdot \sin(x)} + e^x \cdot \cos(x) + e^x \cdot \cos(x) + \cancel{e^x \cdot (-\sin(x))} = 2 \cdot e^x \cdot \cos(x)$ $f''(0) = 2$,

$$f'''(x) = 2\cdot e^x\cdot \cos(x) + 2\cdot e^x\cdot(-\sin(x)) \qquad f'''(x) = 2$$

$$T_3(x) = \underbrace{f(0)}_{0} + \underbrace{f'(0)}_{1}\cdot x + \frac{f''(0)}{\underbrace{2!}_{\frac{2}{2}}}\cdot x^2 + \frac{f'''(0)}{\underbrace{3!}_{\frac{2}{6}}}\cdot x^3 = x + x^2 + \frac{x^3}{3}.$$

$$g(x) = e^x \quad g(0) = 1, \quad g'(x) = e^x \quad g'(0) = 1, \quad g''(x) = e^x \quad g''(0) = 1, \quad g'''(x) = e^x \quad g'''(0) = 1.$$

$$T_3(x) = 1 + 1\cdot x + \frac{1}{2}\cdot x^2 + \frac{1}{6}\cdot x^3$$

$$h(x) = \sin(x) \quad h(0) = 0, \quad h'(x) = \cos(x) \quad h'(0) = 1, \quad h''(x) = -\sin(x) \quad h''(0) = 0,$$

$$h'''(x) = -\cos(x) \quad h'''(0) = -1.$$

$$T_3(x) = 0 + 1\cdot x + 0\cdot x^2 + \frac{-1}{6}x^3.$$

Multiplying:

after removing larger than 3 degrees.

$$\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right) = x - \frac{x^3}{6} + x^2 + \frac{x^3}{2} = x + x^2 + \frac{x^3}{3}.$$