Math 31B
Integration and Infinite Series

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please write your solutions in the space provided below the statement of the problem, box your final answer, show all your work legibly, and clearly reference any theorems or results that you use. Please do not write your solutions in the back of the page, that space should only be used for scratch work. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Once the 50 minutes have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________________
ID number: ___________________________
Section: ____________________________

Question	Points	Score
1	15	
2	17	
3	17	
4	17	
5	17	
6	17	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write **T** in the box provided under the statement. If the statement is false, write **F** in the box provided under the statement. Do not write "true" or "false".

(a) **F** Let S be the solid obtained by rotating the region below a curve $f(x)$. If S has finite volume then S has finite surface area.

(b) **F** Given polynomials $p(x)$ and $q(x)$, then we can always find the partial fraction decomposition of $\frac{p(x)}{q(x)}$.

(c) **T** There are improper integrals that do not converge.

(d) **F** Let $f(x)$ be any function. The nth Taylor polynomial of $f(x)$ coincides with $f(x)$ for n large enough.

(e) **F** The Squeeze Theorem can always be used to determine the convergence of $\lim_{n\to\infty} b_n$ if we know the convergence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} c_n$.

Problem 2. *17pts.*

Find the integral of $f(x) = \frac{3x^2-4x+5}{x^3-x^2+x-1}$ between 2 and 3.

Solution: We have the partial fraction decomposition:

$$\frac{3x^2-4x+5}{x^3-x^2+x-1} = \frac{2}{x-1} + \frac{x-3}{x^2+1}$$

and thus

$$\begin{aligned}\int_2^3 \frac{3x^2-4x+5}{x^3-x^2+x-1}dx &= \int_2^3 \frac{2dx}{x-1} + \int_2^3 \frac{xdx}{x^2+1}dx - \int_2^3 \frac{3dx}{x^2+1} \\ &= 2\ln|x-1|\Big|_2^3 + \frac{1}{2}\ln|x^2+1|\Big|_2^3 - 3\arctan(x)\Big|_2^3 \\ &= \frac{1}{2}\left(\ln(32) + 6\arctan(2) - 6\arctan(3)\right).\end{aligned}$$

Problem 3. *17pts.*

Determine whether $\int_0^\pi \tan(\theta)d\theta$ converges and, if so, evaluate it.

Solution: Note that $\tan(\theta)$ has a vertical asymptote at $\theta = \pi/2$. We then separate the integral as:

$$\int_0^\pi \tan(\theta)d\theta = \int_0^{\pi/2} \tan(\theta)d\theta + \int_{\pi/2}^\pi \tan(\theta)d\theta.$$

Now:

$$\begin{aligned}\int_0^{\pi/2} \tan(\theta)d\theta &= \lim_{N\to\pi/2} \int_0^N \tan(\theta)d\theta \\ &= \lim_{N\to\pi/2} \ln|\sec(\theta)|\Big|_0^N \\ &= \lim_{N\to\pi/2} \ln|\sec(N)| \\ &= \infty.\end{aligned}$$

Since one of the integrals in the expansion of $\int_0^\pi \tan(\theta)d\theta$ is not convergent, then by definition the whole integral also does not converge.

Problem 4. *17pts.*

Compute the surface area of revolution about the x-axis of $f(x) = (4 - x^{2/3})^{3/2}$ in $[0, 8]$.

Solution: We have:

$$f'(x) = \frac{-1}{x^{1/3}}(4 - x^{2/3})^{1/2}$$

so

$$1 + (f'(x))^2 = 1 + \frac{4 - x^{2/3}}{x^{2/3}} = \frac{4}{x^{2/3}}.$$

Hence the surface area of revolution is

$$S = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \frac{2dx}{x^{1/3}}.$$

Setting $u = 4 - x^{2/3}$ then $du = -(2/3)x^{-1/3}dx$ and

$$S = 2\pi \int_0^8 (4 - x^{2/3})^{3/2} \frac{2dx}{x^{1/3}} = 2\pi \int_4^0 (-3)u^{3/2}du = \frac{384}{5}\pi.$$

Problem 5. *17pts.*

(a) Find $T_5(x)$ for $f(x) = \cos(x)$ and evaluate it at $x = 0$.

(b) Find the maximum possible size of the error between $f(x) = \cos(x)$ and $T_5(x)$ around $a = 0$ when evaluated at $x = 0.25$.

Solution:

(a) We have the expansion around $x = 0$:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and thus

$$T_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Hence $T(0) = 1$.

(b) We have $f^{(6)}(x) = -\cos(x)$ so $|f^{(6)}(x)| \leq 1$ for all real numbers x, meaning that we can choose $K = 1$ in the formula to bound the error. Then:

$$|\cos(0.25) - T_5(0.25)| = |f(x) - T_5(x)| \leq \frac{Kx^6}{6!} = \frac{\left(\frac{1}{4}\right)^6}{6!} = \frac{1}{2^{12} \cdot 6!} = \frac{1}{2949120}.$$

Problem 6. *17pts.*

Compute the limit of the sequence with general term $a_n = (2^n + 3^n)^{1/n}$.

Solution: Note that $2^n + 3^n \geq 3^n$ so:

$$(2^n + 3^n)^{1/n} \geq (3^n)^{1/n} = 3.$$

Note that $2^n + 3^n \leq 3^n + 3^n = 2 \cdot 3^n$ so:

$$(2^n + 3^n)^{1/n} \leq (2 \cdot 3^n)^{1/n} = 2^{1/n} \cdot 3.$$

Since

$$\lim_{n\to\infty} 3 = 3 \quad \text{and} \quad \lim_{n\to\infty} 2^{1/n} \cdot 3 = 3$$

then by the Squeeze Theorem for sequences we have

$$\lim_{n\to\infty} (2^n + 3^n)^{1/n} = 3.$$