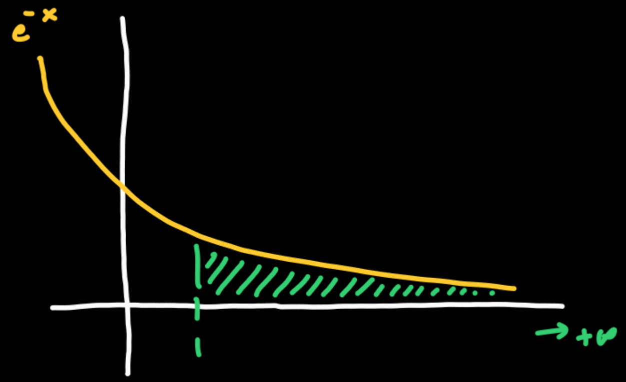
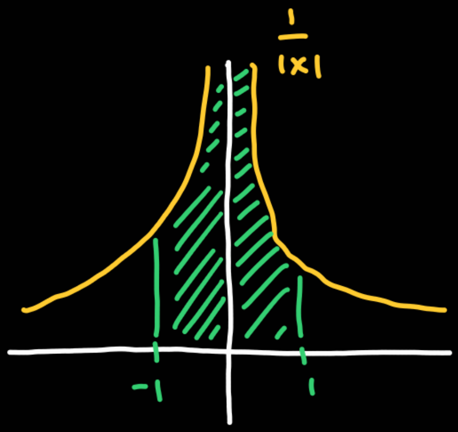
Section 8.6.: Improper integrals.

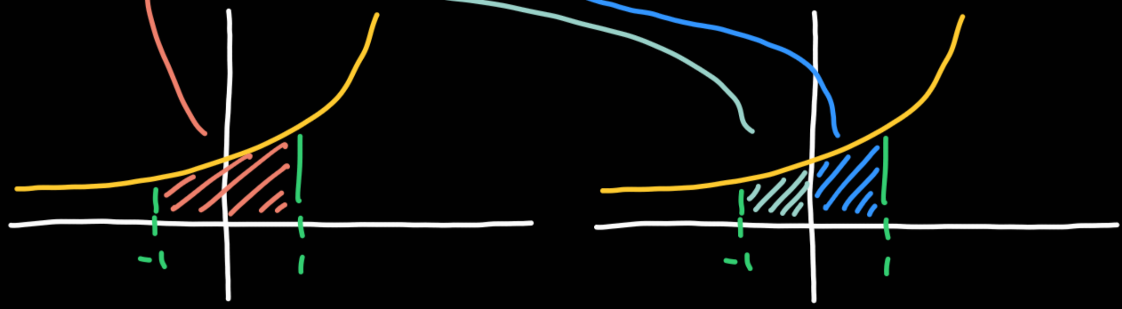
① Infinite interval: $\int_1^{\infty} e^{-x} \cdot dx$.

② Function has discontinuity: $\int_{-1}^{1} \frac{1}{|x|} dx$.



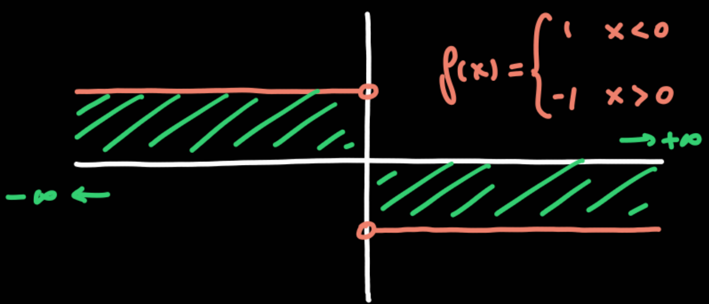


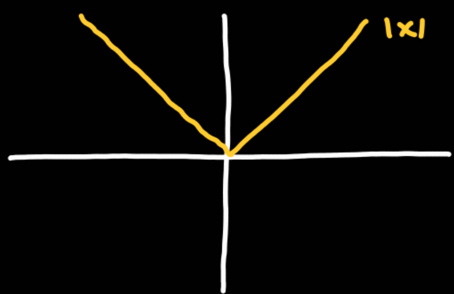
In general: $\underbrace{\int_{-1}^{1} e^{x} \cdot dx}_{} = \underbrace{\int_{-1}^{0} e^{x} \cdot dx}_{} + \underbrace{\int_{0}^{1} e^{x} dx}_{}$



If the integral converges then: $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_{a}^{+\infty} f(x)\, dx$

For $\int_{-\infty}^{\infty} f(x)\, dx$ to converge we need both $\int_{-\infty}^{a} f(x)\, dx$ and $\int_{a}^{+\infty} f(x)\, dx$ to converge.





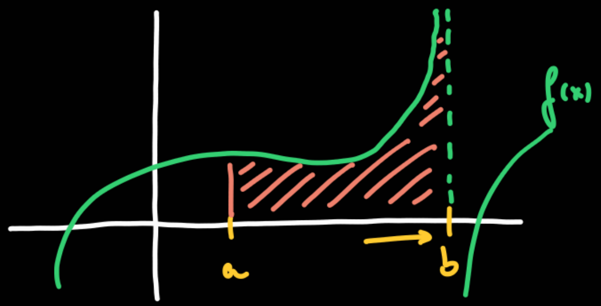
The integral of $f(x)$ from $-\infty$ to $+\infty$ does <u>not</u> converge! ⚠

$$\int_1^\infty e^{-x}\,dx = \underbrace{\lim_{R\to+\infty}}_{\text{compute 2nd}}\Big(\underbrace{\int_1^R e^{-x}\,dx}_{\text{compute 1st}}\Big)$$

real number

If $f(x)$ has a discontinuity at b then:

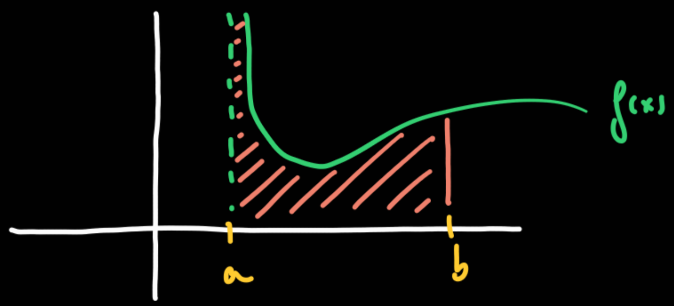
$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to b^-}\int_a^R f(x)\,dx}$$



We replace b by the limit approaching b from the left.

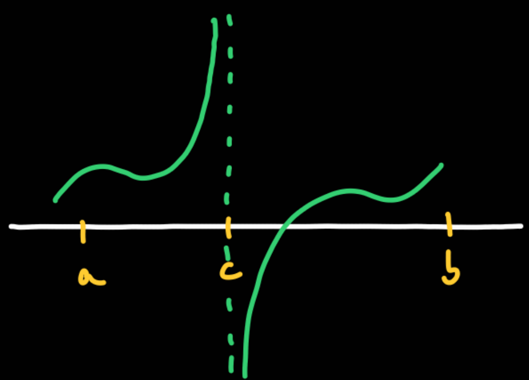
real number

If $f(x)$ has a discontinuity at a then:



$$\boxed{\int_a^b f(x)\,dx = \lim_{R\to a^+}\int_R^b f(x)\,dx}$$

If $f(x)$ has a discontinuity at c, with $a<c<b$:



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

converges when both summands on the right converge.

Examples: $\frac{1}{x}$, $\frac{1}{x^p}$ p real number , ln(x)

discontinuous at 0.

1. $$\int_1^2 \frac{1}{\sqrt{4-x^2}}\cdot dx = \lim_{R\to 2^-}\int_1^R \frac{1}{\sqrt{4-x^2}}\cdot dx = \lim_{R\to 2^-}\int_1^R \frac{1}{\sqrt{4\cdot\left(1-\frac{x^2}{4}\right)}}dx =$$

Careful! $4-2^2=0$, so at $x=2$ we "divide by zero".

The function is discontinuous at $x=2$ and $x=-2$. $4-x^2 = 4\cdot\left(1-\frac{x^2}{4}\right)$

Note: $\frac{1}{\sqrt{4-x^2}}$ resembles $\underbrace{\frac{1}{\sqrt{1-x^2}}}$

we know this integral : arcsin(x)

$$= \lim_{R\to 2^-}\int_1^R \frac{1}{2\cdot\sqrt{1-\frac{x^2}{4}}}\cdot dx = \lim_{R\to 2^-}\int_1^R \frac{1}{2}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot dx =$$

group $\frac{x^2}{4}$ into a single square

$4=2^2$ so $\frac{x^2}{4} = \frac{x^2}{2^2} = \left(\frac{x}{2}\right)^2$

$u = \frac{x}{2}$

$du = \frac{1}{2}\cdot dx$

$u(R) = \frac{R}{2}$

$u(1) = \frac{1}{2}$

$$= \lim_{R\to 2^-}\int_{\frac{1}{2}}^{\frac{R}{2}} \frac{du}{\sqrt{1-u^2}} = \lim_{R\to 2^-} \arcsin(u)\Big|_{\frac{1}{2}}^{\frac{R}{2}} =$$

$$= \lim_{R\to 2^-} \arcsin\left(\frac{R}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \arcsin(1) - \arcsin\left(\frac{1}{2}\right) =$$

$\arcsin(1) = x$ $\sin(\arcsin(1)) = \sin(x)$ $1 = \sin(x)$ $x = \frac{\pi}{2}$

$\arcsin\left(\frac{1}{2}\right) = x$ $\sin\left(\arcsin\left(\frac{1}{2}\right)\right) = \sin(x)$ $\frac{1}{2} = \sin(x)$ $x = \frac{\pi}{6}$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi}{3\cdot 2} - \frac{\pi}{6} = \frac{3\pi - \pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}.$$

2. $\displaystyle\int_2^4 \frac{x}{x^2-9}\cdot dx = \int_2^3 \frac{x\cdot dx}{x^2-9} + \int_3^4 \frac{x\cdot dx}{x^2-9}$

Careful! $x^2-9=0$ has two solutions: $x=\pm 3$.

The discontinuity at $x=3$ bothers us.

$$\int_2^3 \frac{x\cdot dx}{x^2-9} = \lim_{R\to 3^-}\int_2^R \frac{x\cdot dx}{x^2-9} = \lim_{R\to 3^-}\int_{-5}^{R^2-9} \frac{1}{2}\cdot\frac{du}{u} =$$

$u = x^2-9$ $\quad u(R) = R^2-9$ $\quad u(2) = 4-9=-5$

$du = 2x\,dx$ $\quad x\cdot dx = \frac{du}{2}$

$$= \lim_{R\to 3^-} \frac{1}{2}\cdot \ln|u|\Big|_{-5}^{R^2-9} = \lim_{R\to 3^-} \frac{1}{2}\left(\ln|R^2-9| - \ln|-5|\right) =$$

$$= \lim_{R\to 3^-} \frac{1}{2}\cdot\left(\ln|\underbrace{R^2-9}_{\text{goes to 0 when R goes to 3}}| - \ln|5|\right) = -\infty$$

This integral diverges, so the whole integral also diverges.

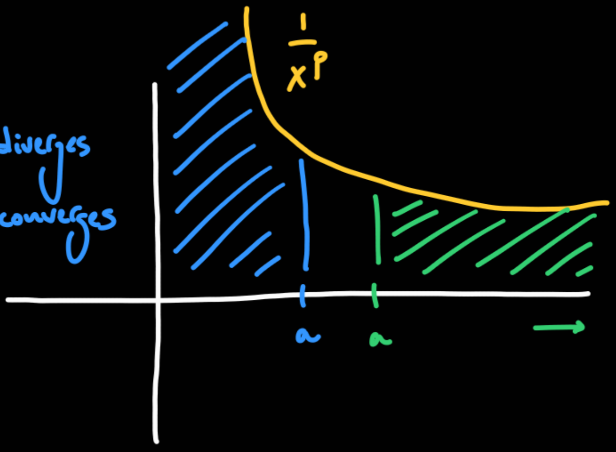
Recall:

p integral from a to ∞: $\quad a>0$ real number

$$\int_a^\infty \frac{1}{x^p}\cdot dx = \begin{cases} \dfrac{a^{1-p}}{p-1} & p>1. \\ \text{diverge} & p\le 1. \end{cases}$$

p-integral from 0 to a: $\quad a>0$ real number

$$\boxed{\int_0^a \frac{1}{x^p}\cdot dx = \begin{cases} \text{diverge} & p\ge 1. \\ \dfrac{a^{1-p}}{1-p} & p<1. \end{cases}}$$



3. $\displaystyle \int_0^1 \frac{1}{x}\,dx = \lim_{R \to 0^+} \int_R^1 \frac{1}{x}\,dx = \lim_{R \to 0^+} \ln|x| \Big|_R^1 = \lim_{R \to 0^+} \underbrace{\ln(1)}_{0} - \underbrace{\ln(R)}_{-\infty} = +\infty$ diverges.

<u>Comparison test:</u> Given $f(x) \geq g(x) \geq 0$, then:

1. If $\int_a^\infty f(x)$ converges then $\int_a^\infty g(x)$ also converges.
2. If $\int_a^\infty g(x)$ diverges then $\int_a^\infty f(x)$ also diverges.

This does <u>not</u> say anything if $\int_a^\infty g(x)$ converges.

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<u>Example:</u> Determine convergence or divergence of $\underbrace{\int_0^\infty e^{-x^2}\,dx}$.

this integral does <u>not</u> have an expression in elementary functions.

Note: $\displaystyle \int_0^\infty e^{-x^2}\,dx = \underbrace{\int_0^1 e^{-x^2}\,dx}_{\text{always finite}} + \underbrace{\int_1^\infty e^{-x^2}\,dx}_{\text{comparison test.}}$

with a p-integral. $(p>1)$

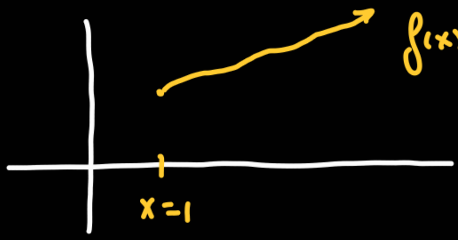
Since exponentials are likely to converge, we compare e^{-x^2} with a converging p-integral. Try comparing with $p=2$: $\frac{1}{x^2}$. Do we have $\frac{1}{x^2} \geq e^{-x^2} \geq 0$ for $x \geq 1$?

? true

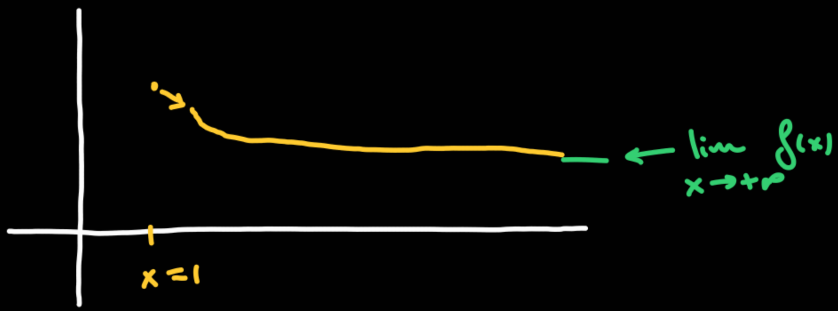
Consider $f(x) = \frac{1}{x^2} - e^{-x^2}$. Is this function positive for $x \geq 1$?

<u>Recall</u>: $f(1) = 1 - e^{-1} = 1 - \frac{1}{e} > 0$.

(1) If $f(x)$ is increasing (i.e. $f'(x) \geq 0$) we are done.



(2) If $f(x)$ decreases, does it cross zero? What is the limit at infinity? If $f(x)$ is decreasing (i.e. $f'(x) \leq 0$ for all $x \geq 1$) and $\lim_{x \to +\infty} f(x) \geq 0$, we are done.



In our case: $f(x) = \frac{1}{x^2} - e^{-x^2}$ so $f'(x) = -\frac{2}{x^3} + 2x e^{-x^2} < 0$.

Because $f'(1) = -2 + \frac{2}{e} < 0$ and $f'(x) = 0$ has <u>no</u> solutions:

$-\frac{2}{x^3} + 2x e^{-x^2} = 0$ $\quad 2x e^{-x^2} = \frac{2}{x^3}$ $\quad e^{-x^2} = \frac{1}{x^4}$ $\quad e^{-y} = \frac{1}{y^2}$

$x \geq 1$ $\quad y = x^2$

$e^y = y^2$ $\quad \frac{1}{e^y} = \frac{1}{y^2}$

Now: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x^2} - e^{-x^2} = 0$, so $f(x)$ is positive.

We know: $\frac{1}{x^2} \geq e^{-x^2} \geq 0$, since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (converging p-integral) then $\int_1^\infty e^{-x^2}\cdot dx$ also converges.

So: $\int_0^\infty e^{-x^2}\cdot dx = \int_0^1 e^{-x^2}\cdot dx + \int_1^\infty e^{-x^2}\cdot dx$ converges.