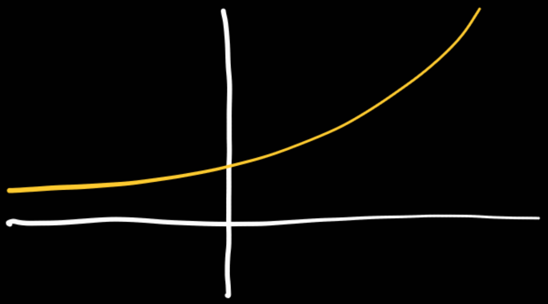
Section 7.1.: Exponential functions, derivatives, and integrals.

$f(t) = b^t$ ← base , positive numbers. $(b>0)$

$f(x) = b^x$ ← input , real numbers.



1. Always strictly positive.
2. It is strictly increasing. $b>1$.
3. It is strictly decreasing if $b<1$.

Laws of exponents:

$$b^0 = 1$$

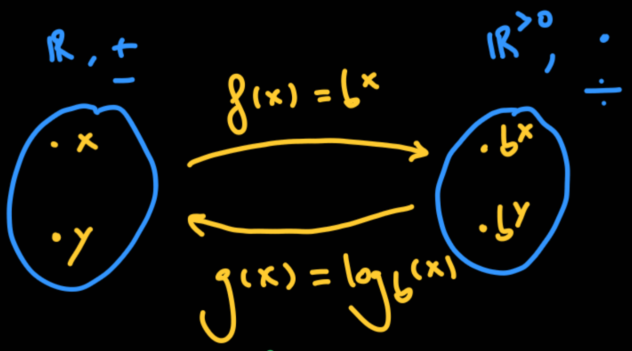
$$b^{x+y} = b^x \cdot b^y$$

$$b^{x-y} = \frac{b^x}{b^y}$$

$$b^{-y} = \frac{1}{b^y}$$

$$(b^x)^y = b^{xy}$$

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$



the exponential function is invertible:

it is completely determined by its input

AND its output.

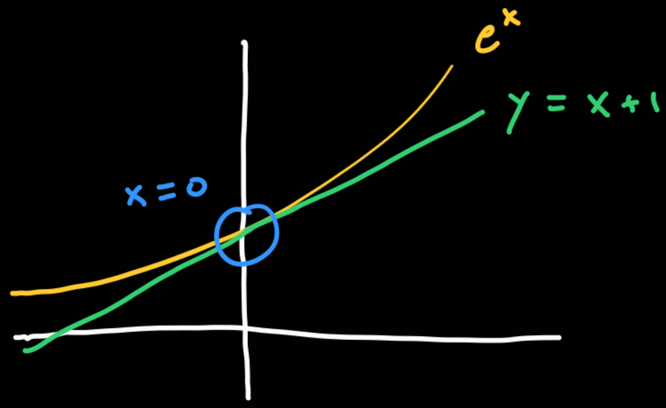
$$27^{\frac{2}{3}} = 9$$

f, g are inverses of each other if $f(g(y)) = y$ AND $g(f(x)) = x$.

Derivative: $f'(x)$ $\frac{d}{dx} f(x)$ is proportional to $f(x)$ when $f(x)$ is an exponential.

$$\boxed{\frac{d}{dx}(b^x) = \ln(b)\cdot b^x}$$

ln the natural logarithm (base e).



At $x=0$ the slope of the line tangent to e^x is 1.

Example: Find the equation of the tangent line to $\underbrace{3e^x + 5x^2}_{f(x)}$ at $x = \underset{7}{2}$.

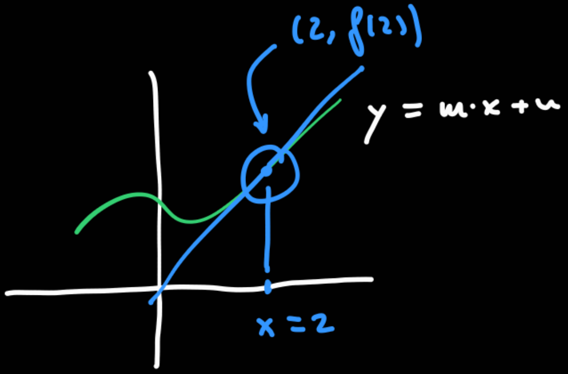
$$\frac{d}{dx} f(x) = 3\cdot\frac{d}{dx} e^x + 5\cdot\frac{d}{dx} x^2 = 3\cdot e^x + 10\cdot x$$

The slope of the tangent line is:

$x = 2 \rightsquigarrow 3e^2 + 20$.

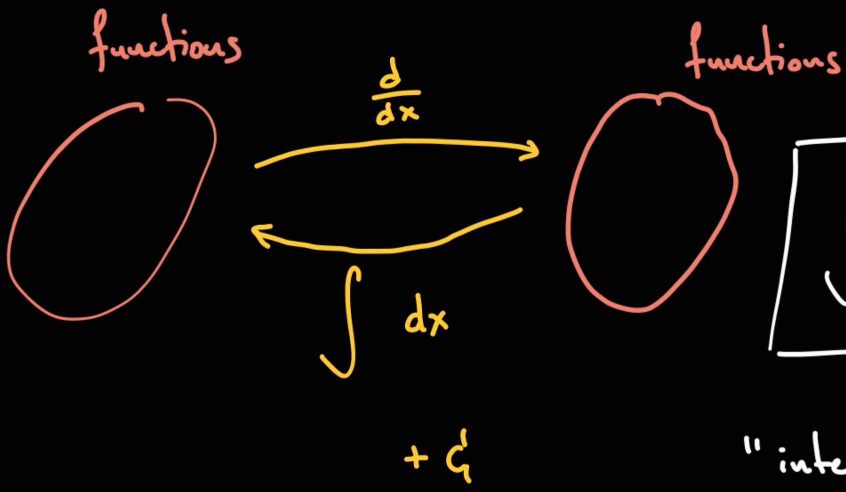
$f(2) = 3e^2 + 20$.

$y = f(\overset{7}{2}) + f'(\overset{7}{2})\cdot(x - \overset{7}{2})$



Chain rule :

$$\boxed{\frac{d}{dx}\left(e^{g(x)}\right) = g'(x)\cdot e^{g(x)}.}$$



$$\boxed{\int g'(x)\cdot e^{g(x)}\cdot dx = e^{g(x)}} + c_1$$

"integration by parts"

Examples:

1) $\frac{d}{dx}\left(e^{\cos(x)}\right) = -\sin(x)\cdot e^{\cos(x)}$

2) $\int x\cdot e^{2x^2}\cdot dx = \frac{1}{4}\cdot e^{2x^2} + c_1'$

$$= \int \frac{4x}{4}\cdot e^{2x^2}\cdot dx = \frac{1}{4}\int 4x\cdot e^{2x^2}\cdot dx = \frac{1}{4}e^{2x^2} + c_1.$$

Derivation and integration are "operations":

$$\frac{d}{dx}\left(\underset{\substack{\uparrow\\ \text{real}\\ \text{number}}}{a}\cdot f(x) + \underset{\substack{\uparrow\\ \text{real}\\ \text{number}}}{b}\cdot g(x)\right) = a\cdot\left(\frac{d}{dx}f(x)\right) + b\cdot\left(\frac{d}{dx}g(x)\right)$$

"derivation is a linear operation"

$$\int \left(a\cdot f(x) + b\cdot g(x)\right)dx = a\cdot\left(\int f(x)dx\right) + b\cdot\left(\int g(x)\,dx\right)$$

"integration is a linear operation"

$$3)\ \int \frac{e^t}{1+2e^t+e^{2t}}\,dt = \int \frac{du}{u^2} = -(u)^{-1} + c_1' = -(e^t+1)^{-1} + c_1 .$$

$$(1+e^t)^2$$

$$u = e^t + 1$$

$$du = e^t\,dt$$

$$u = e^t$$

$$du = e^t\,dt$$