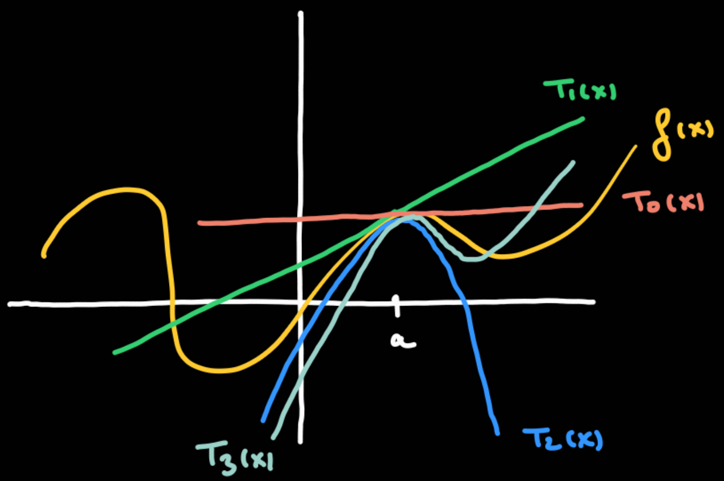
Section 9.4.: Taylor polynomials

Polynomials are easy to work with. Given a complicated function $f(x)$, we want to approximate it using polynomials.

The n-th Taylor polynomial approximates $f(x)$ using the first n derivatives.

$$\boxed{T_n(x) = f(a) + \frac{f'(a)}{1!}\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n}$$

(around a)



$$T_n(x) + \frac{f^{(n+1)}(a)}{(n+1)!}\cdot(x-a)^{n+1} = T_{n+1}(x)$$

The derivatives of $T_n(x)$ coincide with the derivatives of $f(x)$ at a.

less than or equal to n derivatives

Example: Compute the Taylor polynomial of degree 3 and 4 of $f(x) = e^{-2x}$ around 0.

$f'(x) = -2\cdot e^{-2x}$ $\quad f''(x) = 4\cdot e^{-2x}$ $\quad f'''(x) = -8\cdot e^{-2x}$ $\quad f^{IV}(x) = 16\cdot e^{-2x}$

$f(0) = 1$ $\quad f'(0) = -2$ $\quad f''(0) = 4$ $\quad f'''(0) = -8$ $\quad f^{IV}(0) = 16$

$n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1$

$$T_3(x) = f(0) + \frac{f'(0)}{1!}\cdot(x-0) + \frac{f''(0)}{2!}\cdot(x-0)^2 + \frac{f'''(0)}{3!}\cdot(x-0)^3 =$$

$$= 1 - 2x + 2x^2 + \frac{(-8)}{6}\cdot x^3 = 1-2x+2x^2 - \frac{4}{3}x^3.$$

$$T_4(x) = \underbrace{f(0) + \frac{f'(0)}{1!}\cdot(x-0) + \frac{f''(0)}{2!}\cdot(x-0)^2 + \frac{f'''(0)}{3!}\cdot(x-0)^3}_{T_3(x)} + \underbrace{\frac{f^{IV}(0)}{4!}\cdot(x-0)^4}_{} =$$

$$= 1-2x+2x^2 - \frac{4}{3}x^3 + \frac{16}{24}\cdot x^4 = 1-2x+2x^2-\frac{4}{3}x^3 + \frac{2}{3}x^4.$$

Sometimes you will be asked for the general term: compute $\frac{f^{(n)}(0)}{n!}\cdot(x-0)^n$

$$f^{(n)}(x) = (-1)^n\cdot 2^n\cdot e^{-2x} \qquad f^{(n)}(0) = (-1)^n\cdot 2^n \qquad \frac{(-1)^n\cdot 2^n}{n!}\cdot x^n$$

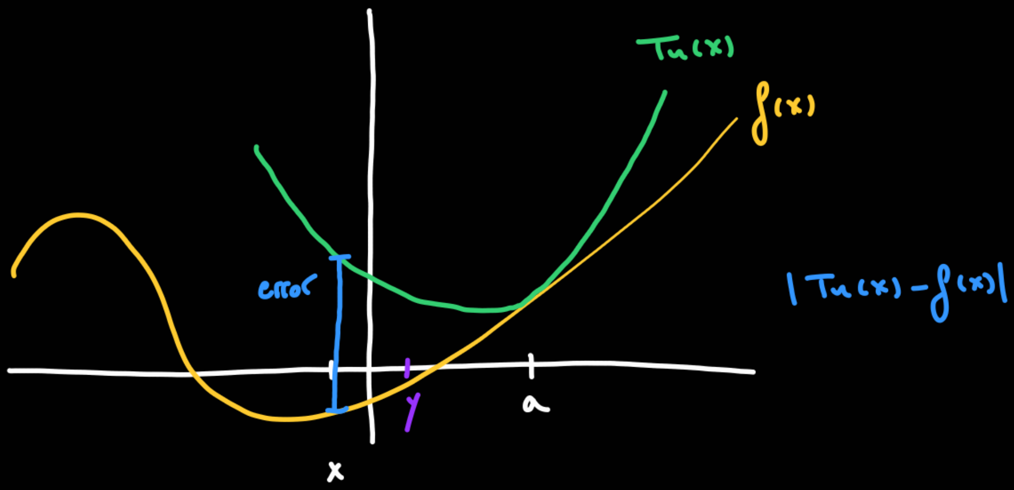
Useful Taylor polynomials to know:

e^x	0	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$	
$\sin(x)$	0	$x - \frac{x^3}{3!} + \dots + (-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$	$T_{2n+2}(x) = T_{2n+1}(x)$
$\cos(x)$	0	$1 - \frac{x^2}{2!} + \dots + (-1)^n\cdot\frac{x^{2n}}{(2n)!}$	$T_{2n}(x) = T_{2n+1}(x)$
$\ln(x)$	1	$(x-1) - \frac{1}{2}\cdot(x-1)^2 + \frac{1}{3}\cdot(x-1)^3 - \dots + \frac{(-1)^{n-1}}{n}\cdot(x-1)^n$	
$\frac{1}{1-x}$	0	$1 + x + x^2 + x^3 + \dots + x^n$	

Approximating by Taylor polynomials has the error bounded. $f(x)$, $T_n(x)$, around a

$$\boxed{|T_n(x) - f(x)| \leq K\cdot\frac{|x-a|^{n+1}}{(n+1)!}}$$

with K real constant such that $|f^{(n+1)}(y)| \leq K$ for all y between a and x.



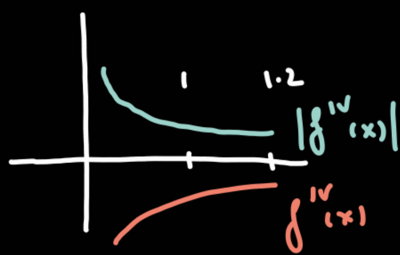
Choose K such that $\left|f^{(n+1)}(y)\right| \leq K$.

<u>Problem 1:</u> Compute the error bound: Find the error bound for $f(x) = \ln(x)$ and $T_3(x)$ around $a = 1$ when looking at $x = 1.2$.

$$|T_3(1.2) - f(1.2)| \leq K \cdot \frac{|1.2-1|^4}{24}$$

We know that K is such that $\left|f^{IV}(y)\right| \leq K$ for all y between 1 and 1.2.

$$f^{IV}(x) = -6 \cdot \frac{1}{x^4}$$



$$\left|f^{IV}(x)\right| = 6 \cdot \frac{1}{x^4}$$

The largest value of $\left|f^{IV}(x)\right|$ is at $x=1$. Since $\left|f^{IV}(1)\right| = 6$, and we need: $\left|f^{IV}(y)\right| \leq 6 = \left|f^{IV}(1)\right| \leq K$, chose $K = 6$.

So the error bound is:

$$|T_3(1.2) - f(1.2)| \leq 6 \cdot \frac{|1.2-1|^4}{24} = \frac{0.2^4}{4} = \frac{\left(\frac{1}{5}\right)^4}{4} = \frac{1}{5^4 \cdot 4} = \frac{1}{5^2 \cdot 5 \cdot 5 \cdot 4} = \frac{1}{2500}.$$

Problem 2: Given a function and a point and an error, find the n-Taylor polynomial satisfying that the error is less than what we are given.

Consider $\cos(x)$ around $a=0$, point $x=0.2$, find n such that $|T_n(x)-f(x)|<10^{-5}$.

Since $|f^{(n)}(x)|=|\cos(x)|$ for n even and $|f^{(n)}(x)|=|\sin(x)|$ for n odd, we always have $|f^{(n)}(x)|\leq 1$. Take $K=1$.

The error bound gives:

$$|T_n(0.2)-\cos(0.2)| \leq 1\cdot \frac{|0.2-0|^{n+1}}{(n+1)!} = \frac{0.2^{n+1}}{(n+1)!} < \frac{1}{100000}$$

find n so that this happens.

n	2	3	4
$\frac{1}{5^{n+1}\cdot(n+1)!}=\frac{0.2^{n+1}}{(n+1)!}$	$\frac{1}{5^3\cdot 3!}=\frac{1}{750}$	$\frac{1}{5^4\cdot 4!}=\frac{1}{15000}$	$\frac{1}{5^5\cdot 5!}=\underbrace{\frac{1}{375000}}$

this is smaller than $\frac{1}{10000}$

So $n=4$.