Note: Other potential solutions exists

Math 31B LA Review Solutions

7.3 #40

$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

$$y = \ln(\cot(x))$$

$$y' = \frac{1}{\cot(x)} \cdot \frac{d}{dx}(\cot(x)) = \frac{1}{\cot(x)} \cdot -\csc^2(x) = \frac{\sin(x)}{\cos(x)} \cdot \frac{-1}{\sin^2(x)}$$

$$\frac{d}{dx}\left[\ln(\cot(x))\right] = \boxed{\frac{-1}{\sin(x)\cos(x)}}$$

7.7 #28

$$\lim_{x\to 0}\left(\cot(x) - \frac{1}{x}\right)$$

*note: got rid of $\cot(x)$ because $\lim_{x\to 0} \cot(x) = \text{DNE}$

$$= \lim_{x\to 0} \frac{x\cot(x) - 1}{x}^{*} = \lim_{x\to 0} \frac{x\left(\frac{\cos(x)}{\sin(x)}\right) - 1}{x} = \lim_{x\to 0} \frac{x\cos(x) - \sin(x)}{x\sin(x)} \begin{matrix} \to 0 \\ \to 0 \end{matrix} \quad \text{LHR}$$

$$= \lim_{x\to 0} \frac{\cos(x) + x(-\sin(x)) - \cos(x)}{x\cos(x) + \sin(x)} = \lim_{x\to 0} \frac{-x\sin(x)}{x\cos(x) + \sin(x)} \begin{matrix} \to 0 \\ \to 0 \end{matrix} \quad \text{LHR}$$

$$= \lim_{x\to 0} \frac{-x\cos(x) + -\sin(x)}{x(-\sin(x)) + \cos(x)} = \frac{-0\cdot 1 - 0}{0\cdot 0 + 1} = \frac{0}{1} = 0$$

$$\therefore \boxed{\lim_{x\to 0} \cot(x) - \frac{1}{x} = 0}$$

7.7 #55

$f(x) = x^{1/x}$ for $x > 0$

a) Calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$

$$\lim_{x\to 0^+} x^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln(x)^{\frac{1}{x}}} = \lim_{x\to 0^+} e^{\frac{1}{x}\ln(x)} = \lim_{x\to 0^+} e^{\frac{\ln(x)}{x}}$$

$$= e^{\frac{-\infty}{0}} = e^{-\infty} = \frac{1}{e^{\infty}} = 0 \Rightarrow \boxed{\lim_{x\to 0^+} x^{1/x} = 0}$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln(x)^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} = e^{\lim_{x\to\infty}\frac{\ln(x)}{x}}$$

only looking at limit

$$\lim_{x\to\infty} \frac{\ln(x) \to \infty}{x \to \infty} \text{ LHR } \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$$e^{\lim_{x\to\infty}\frac{\ln(x)}{x}} = e^0 = \boxed{1}$$

b) $y = x^{1/x}$

$\ln(y) = \frac{1}{x}\ln(x)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{\ln(x)}{x^2} \Rightarrow \frac{dy}{dx} = y\left(\frac{1-\ln(x)}{x^2}\right) \Rightarrow f'(x) = x^{1/x}\left(\frac{1-\ln(x)}{x^2}\right)$$

Solving for $f'(x) = 0$, we find $\ln(x) = 1$ or $x = e$ and $x = 0$

However, $x > 0$ for $f(x)$, thus left with $x = e$ only. $f(e) = e^{1/e}$

plugging in value above e and less than e into $f'(x)$

$f'(1) = 1^1\left(\frac{1}{1}\right) = 1$

positive means increasing

$$f'(e^2) = (e^2)^{\frac{1}{e^2}}\frac{1 - 2\ln(e)}{e^2} = e^{\frac{2}{e^2}}\frac{1-2}{e^2} = \frac{-e^{\frac{2}{e^2}}}{e^2}$$

negative means decreasing

$\boxed{\text{Thus, } f(x) \text{ is increasing } 0 < x < e \text{ and decreasing } x > e}$

7.7 #64

Let $H(b) = \lim\limits_{x\to\infty} \frac{\ln(1+b^x)}{x}$ for $b>0$

a) Show that $H(b) = \ln(b)$ if $b \geq 1$

$$\lim_{x\to\infty} \frac{\ln(1+b^x) \to \infty}{x \quad \to \infty} \quad \text{LHR} \quad \lim_{x\to\infty} \frac{\frac{1}{1+b^x} \cdot \ln(b) \cdot b^x}{1} = \lim_{x\to\infty} \frac{\ln(b)\cdot b^x}{1+b^x}$$

$$= \lim_{x\to\infty} \frac{\ln(b)}{b^{-x}+1} = \frac{\ln(b)}{0+1} = \boxed{\ln(b)}$$

b) when $0<b\leq 1$

$$\lim_{x\to\infty} \frac{\ln(1+b^x)}{x} \Rightarrow \lim_{x\to\infty} \ln(1+0^x) \leq \lim_{x\to\infty} \ln(1+b^x) \leq \lim_{x\to\infty} \ln(1+1^x)$$

$$0 \leq \lim_{x\to\infty} \ln(1+b^x) \leq \ln(2)$$

$$\lim_{x\to\infty} \frac{\ln(1+0^x)}{x} \leq \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} \leq \lim_{x\to\infty} \frac{\ln(1+1^x)}{x}$$

$$\lim_{x\to\infty} \frac{0}{x} \leq \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} \leq \lim_{x\to\infty} \frac{\ln(2)}{x}$$

$$0 \leq \lim_{x\to\infty} \frac{\ln(1+b^x)}{x} \leq 0$$

Thus, by squeeze theorm, $\boxed{\lim\limits_{x\to\infty} \frac{\ln(1+b^x)}{x} = 0}$

Note:

$\lim\limits_{x\to\infty} 0^x = 0$ because it is 0 not approaching 0

7.9 #60

$$\int_{-3}^{-1} \frac{dx}{x\sqrt{x^2+16}} = \int_{-3}^{-1} \frac{1}{x\sqrt{16\left(\frac{x^2}{16}+1\right)}}\,dx = \int_{-3}^{-1} \frac{1}{4x\sqrt{\left(\frac{x}{4}\right)^2+1}}\,dx$$

$u = \frac{x}{4}$

$\frac{du}{dx} = \frac{1}{4}$

$4\,du = dx$

$$\int_a^b \frac{1}{16u\sqrt{u^2+1}}\,4\,du = \frac{1}{4}\int_a^b \frac{1}{u\sqrt{u^2+1}}\,du$$

we see this is similar to inverse hyperbolic function $\operatorname{csch}^{-1}(x)$

where $\frac{d}{dx}\operatorname{csch}^{-1}(x) = \frac{-1}{|x|\sqrt{x^2+1}}$

for u, we can solve for a and b.

$a = \frac{-3}{4}$ $\qquad$ $b = \frac{-1}{4}$

So, $\frac{1}{4}\int_{-3/4}^{-1/4} \frac{1}{u\sqrt{u^2+1}}$

we can see this is the form of $\operatorname{csch}^{-1}(x)$ when x is negative

Thus, $$\frac{1}{4}\left[\operatorname{csch}^{-1}(u)\right]_{-3/4}^{-1/4} = \boxed{\frac{1}{4}\left[\operatorname{csch}^{-1}\left(\frac{-1}{4}\right) - \operatorname{csch}^{-1}\left(\frac{-3}{4}\right)\right]}$$

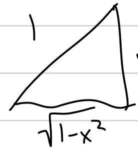
8.1 #67

$$\int \left(\sin^{-1}(x)\right)^2 dx$$

Integration by Parts:

$u = \left(\sin^{-1}(x)\right)^2 \quad v = x$

$du = \dfrac{2 \cdot \sin^{-1}(x)}{\sqrt{1-x^2}} \quad dv = 1$



$$= x\left(\sin^{-1}(x)\right)^2 - \int \frac{2x\sin^{-1}(x)}{\sqrt{1-x^2}}\, dx$$

$u = \sin^{-1}(x)$

$du = \dfrac{1}{\sqrt{1-x^2}}\, dx$

$$= x\left(\sin^{-1}(x)\right)^2 - \int \frac{2\sin(u)\cdot u}{\cancel{\sqrt{1-x^2}}} \cdot \cancel{\sqrt{1-x^2}}\, du$$

$$= x\left(\sin^{-1}(x)\right)^2 - \int 2u\sin(u)\, du$$

$2u$	$-\cos(u)$
2	$\sin(u)$

$$= x\left(\sin^{-1}(x)\right)^2 - \left((2u)(-\cos(u)) + 2\int \cos(u)\, du\right)$$

$$= x\left(\sin^{-1}(x)\right)^2 + 2u\cos(u) - 2\sin(u) + C$$

$$\boxed{= x\left(\sin^{-1}(x)\right)^2 + 2\sin^{-1}(x)\sqrt{1-x^2} - 2x + C}$$

8.1 #68

$$\int \frac{(\ln(x))^2}{x^2}\,dx \qquad t=\ln(x) \qquad e^t = x$$

$$\frac{dt}{dx} = \frac{1}{x} \qquad x\,dt = dx$$

$$= \int \frac{t^2}{x^2}\,x\,dt = \int \frac{t^2}{x}\,dt = \int \frac{t^2}{e^t}\,dt$$

Integration by parts

$U = t^2 \quad V = -e^{-t} \qquad du = 2t \quad dv = e^{-t}$

$$= \int t^2 e^{-t}\,dt = -t^2 e^{-t} - \int 2te^{-t}\,dt$$

IBP $\quad a = 2t \quad b = -e^{-t}$

$\quad da = 2 \quad db = e^{-t}$

$$= -t^2 e^{-t} - 2te^{-t} + 2\int e^{-t}\,dt$$

$$= -\left(t^2 e^{-t} + 2te^{-t} + 2e^{-t}\right) + C$$

resubbing in from t-sub

$$\boxed{= (-1)\left[\frac{(\ln x)^2 + 2\ln(x) + 2}{x}\right] + C}$$

8.5 #39

$$\int \frac{1}{x(x^2+8)^2}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+8} + \frac{Dx+E}{(x^2+8)^2}\,dx$$

what to multiply by to get to original denominator

$$\cdot\frac{(x^2+8)^2}{(x^2+8)^2} \qquad \cdot\frac{x(x^2+8)}{x(x^2+8)} \qquad \cdot\frac{x}{x}$$

$$A(x^2+8)^2 = Ax^4+16Ax^2+64A$$

$$(Bx+C)(x^2+8)(x) = Bx^4+Cx^3+8Bx^2+8Cx$$

$$(Dx+E)(x) = Dx^2+Ex$$

we know $Ax^4+16Ax^2+64A+Bx^4+Cx^3+8x^2+8Cx+Dx^2+Ex = 1$

x^0	$64A=1$		$A=\frac{1}{64}$
x^1	$E+8C=0$	After solving	$B=\frac{-1}{64}$
x^2	$D+16A+8B$	$\rightarrow$	$C=0$
x^3	$C=0$		$D=\frac{-1}{4}+\frac{1}{8}=\frac{-1}{8}$
x^4	$A+B=0$		$E=0$

$$\int \frac{1}{64x}\,dx + \int \frac{-x}{64(x^2+8)}\,dx + \int \frac{-x}{8(x^2+8)^2}\,dx$$

$u=x^2+8 \quad \frac{du}{dx}=2x$

$$\frac{1}{64}\int \frac{1}{x}\,dx + \frac{-1}{64}\int \frac{x}{x^2+8}\,dx + \frac{-1}{8}\int \frac{\cancel{x}}{u^2}\frac{du}{2\cancel{x}}$$

$u=x^2+8 \quad \frac{du}{dx}=2x$

$$-\frac{1}{64}\int \frac{\cancel{x}}{u}\frac{du}{2\cancel{x}} + \frac{-1}{8}\cdot\frac{1}{2}\int u^{-2}\,du$$

$$-\frac{1}{64}\cdot\frac{1}{2}\int \frac{1}{u}\,du + \frac{-1}{16}\,\frac{u^{-1}}{-1}$$

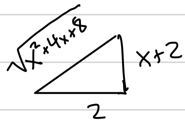
$$\boxed{=\frac{1}{64}\ln|x| - \frac{1}{128}\ln|x^2+8| + \frac{1}{16(x^2+8)} + C}$$

used

* $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ * $\tan^2\theta + 1 = \sec^2\theta$

8.5 #64

$\tan(\theta) = \frac{x+2}{2}$



$$\int \frac{(x+1)\,dx}{(x^2+4x+8)^2} = \int \frac{(x+2)\ -1}{(x^2+4x+8)^2}\,dx$$

$$= \int \frac{(x+2)}{(x^2+4x+8)^2}\,dx - \int \frac{1}{(x^2+4x+8)^2}\,dx$$

First integral $u = x^2+4x+8$

$\frac{du}{dx} = 2x+4$

$$= \int \frac{\cancel{(x+2)}}{u^2}\,\frac{du}{2\cancel{(x+2)}} = \frac{1}{2}\int \frac{1}{u^2}\,du = \frac{1}{2}\ \frac{-1}{u} = \frac{-1}{2(x^2+4x+8)}$$

Second integral

$$= \int \frac{1}{(x^2+4x+8)^2}\,dx = \int \frac{1}{((x^2+4x+4)+4)^2}\,dx = \int \frac{1}{((x+2)^2+4)^2}\,dx \qquad v = x+2,\ dv = dx$$

$$= \int \frac{1}{(v^2+4)^2} \qquad 2z = v,\ 2\,dz = dv \qquad = \int \frac{2}{(4(z^2+1))^2}\,dz = \frac{1}{8}\int \frac{1}{(z^2+1)^2}\,dz \qquad z = \tan a,\ \frac{dz}{da} = \sec^2 a$$

$$= \frac{1}{8}\int \frac{\sec^2 a}{(\tan^2 a+1)^2}\,da = \frac{1}{8}\int \frac{\cancel{\sec^2 a}}{\sec^{4\,2} a}\,da = \frac{1}{8}\int \cos^2(a)\,da$$

$$= \frac{1}{16}\int 1+\cos(2a)\,da = \frac{1}{16}\,a + \frac{1}{32}\sin(2a) \Rightarrow \frac{1}{16}\tan^{-1}(z) + \frac{1}{32}\sin(2\tan^{-1}(z))$$

$$\Rightarrow \frac{1}{16}\tan^{-1}\left(\frac{v}{2}\right) + \frac{1}{32}\sin\left(2\tan^{-1}\left(\frac{v}{2}\right)\right) \Rightarrow \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{1}{32}\sin\left(2\tan^{-1}\left(\frac{x+2}{2}\right)\right)$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{1}{32}\left(2\sin\left(\tan^{-1}\left(\frac{x+2}{2}\right)\right)\cos\left(\tan^{-1}\left(\frac{x+2}{2}\right)\right)\right)$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) + \frac{1}{8}\left(\frac{(x+2)}{(x^2+4x+8)}\right)$$

Together, we get

$$\boxed{= \frac{-1}{16}\tan^{-1}\left(\frac{x+2}{2}\right) - \frac{1}{8}\left(\frac{x+2}{x^2+4x+8}\right) - \frac{1}{2(x^2+4x+8)} + C}$$