A variant of Problem 11.3.39.: Prove convergence or divergence of $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$.

First, we make an educated guess as to whether the series converges or diverges. When n is very large n^3-1 behaves like n^3, so $\frac{n}{n^3-1}$ behaves like $\frac{n}{n^3} = \frac{1}{n^2}$, so $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ behaves like $\sum_{n=2}^{\infty} \frac{1}{n^2}$, a converging p-series.

Second, we show $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ converges.

Method 1: Using the Limit Comparison Test. Let $a_n = \frac{n}{n^3-1}$, $b_n = \frac{1}{n^2}$, since:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^3-1} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1$$

and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is a converging p-series, then $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ also converges.

Method 2: Using the Integral Test. We compute $\int_2^{\infty} \frac{x}{x^3-1}\,dx$. To do this, we decompose $\frac{x}{x^3-1}$ into partial fractions. Note that $x=1$ is a root of x^3-1 because $(1)^3-1=0$, so $x-1$ divides x^3-1. Long division (the European way) gives:

$$\begin{array}{rl|l}
 & x^3-1 & x-1 \\ \hline
- & x^3-x^2 & x^2+x+1 \\ \hline
 & 0+x^2-1 & \\
- & x^2-x & \\ \hline
 & 0+x-1 & \\
- & x-1 & \\ \hline
 & 0 &
\end{array}$$

so $x^3-1 = (x-1)\cdot(x^2+x+1)$.

The factor $x-1$ cannot be decomposed any further. The roots of x^2+x+1 are:

$$x = \frac{-1 \pm \sqrt{1^2-4\cdot1\cdot1}}{2\cdot1} = \frac{-1\pm\sqrt{-3}}{2}$$

, which are <u>not</u> real numbers.

Then, the factor x^2+x+1 cannot be decomposed further. The partial fraction decomposition looks as follows:

$$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

Taking common denominator:

$$\frac{x}{x^3-1} = \frac{A\cdot(x^2+x+1)+(Bx+C)\cdot(x-1)}{(x-1)\cdot(x^2+x+1)}$$

so removing denominators we have the equality $x = A\cdot(x^2+x+1)+(Bx+C)\cdot(x-1)$.

Substituting $x=1$ gives:

$$1 = A\cdot(1+1+1)+(Bx+C)\cdot(1-1) \longrightarrow 1 = 3\cdot A \longrightarrow A = \frac{1}{3}.$$

Replacing $A=\frac{1}{3}$ gives the equality $x = \frac{1}{3}\cdot(x^2+x+1)+(Bx+C)\cdot(x-1)$. Substituting $x=0$ gives:

$$0 = \frac{1}{3}\cdot(0+0+1)+(B\cdot 0+C)\cdot(0-1) \longrightarrow 0 = \frac{1}{3}-C \longrightarrow C = \frac{1}{3}.$$

Replacing $C=\frac{1}{3}$ gives the equality $x = \frac{1}{3}\cdot(x^2+x+1)+(Bx+\frac{1}{3})\cdot(x-1)$. Substituting $x=-1$ gives:

$$-1 = \frac{1}{3}\cdot(1-1+1)+(B\cdot(-1)+\frac{1}{3})\cdot(-1-1) \longrightarrow -1 = \frac{1}{3}-2\cdot B-\frac{2}{3} \longrightarrow B = \frac{-1}{3}.$$

The resulting partial fraction decomposition is:

$$\frac{x}{x^3-1} = \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1} = \frac{1}{3}\cdot\left(\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right).$$

We then rewrite $\int_2^\infty \frac{x}{x^3-1}\,dx = \int_2^\infty \frac{1}{3}\cdot\left(\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right)dx$. We now compute:

$$\int_2^\infty \frac{1}{3}\cdot\left(\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right)dx = \lim_{R\to\infty} \int_2^R \frac{1}{3}\cdot\left(\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right)dx =$$

$$= \lim_{R\to\infty}\left(\frac{1}{3}\int_2^R \frac{dx}{x-1} + \frac{1}{3}\int_2^R \frac{1-x}{x^2+x+1}\,dx\right).$$

We now compute $\int_2^R \frac{dx}{x-1}$ and $\int_2^R \frac{1-x}{x^2+x+1}\,dx$.

$$\int_2^R \frac{dx}{x-1} = \ln|x-1|\Big|_2^R = \ln|R-1| - \ln|2-1| = \ln|R-1|.$$

To compute $\int_2^R \frac{1-x}{x^2+x+1}\,dx$ we force the derivative of the denominator to appear in the numerator. The derivative of x^2+x+1 is $2x+1$, so we want to rewrite $\frac{1-x}{x^2+x+1}$ as a sum of $\frac{2x+1}{x^2+x+1}$ and $\frac{1}{x^2+x+1}$, both of which we can integrate. Now we want to find real numbers α and β such that:

$$\frac{1-x}{x^2+x+1} = \alpha\cdot\frac{2x+1}{x^2+x+1} + \beta\cdot\frac{1}{x^2+x+1}, \text{ so:}$$

$$\frac{1-x}{x^2+x+1} = \frac{\alpha\cdot(2x+1)+\beta}{x^2+x+1},$$ which will happen when the numerators coincide:

$1-x = \alpha\cdot(2x+1)+\beta$, and expanding this into an equality of polynomials gives:

$-x+1 = (2\cdot\alpha)\cdot x + (\alpha+\beta)$.

The coefficients of x^1 should be equal: $-1 = 2\cdot\alpha$. The coefficients of x^0 should be equal: $1 = \alpha + \beta$. The system of equations:

$$\begin{cases} -1 = 2\alpha \\ 1 = \alpha + \beta \end{cases} \quad \text{has solution} \quad \begin{cases} \alpha = \frac{-1}{2}, \\ \beta = \frac{3}{2}. \end{cases}$$

We can thus rewrite:

$$\frac{1-x}{x^2+x+1} = \frac{-1}{2} \cdot \frac{2x+1}{x^2+x+1} + \frac{3}{2} \cdot \frac{1}{x^2+x+1}$$

and we can decompose the integral as:

$$\int_2^R \frac{1-x}{x^2+x+1} dx = \frac{-1}{2} \cdot \int_2^R \frac{2x+1}{x^2+x+1} dx + \frac{3}{2} \cdot \int_2^R \frac{1}{x^2+x+1} \cdot dx.$$

Now:

$$\int \frac{2x+1}{x^2+x+1} \cdot dx \underset{\substack{u = x^2+x+1 \\ du = (2x+1)dx}}{=} \int \frac{du}{u} = \ln|u| = \ln|x^2+x+1| \quad \text{, so:}$$

$$\int_2^R \frac{2x+1}{x^2+x+1} \cdot dx = \ln|x^2+x+1| \Big|_2^R = \ln|R^2+R+1| - \ln|7|.$$

To compute $\int \frac{1}{x^2+x+1} \cdot dx$ we complete the square to rewrite the denominator:

$\frac{1}{x^2+x+1} = \frac{1}{(x+a)^2+b}$ occurs when $x^2+x+1 = (x+a)^2+b$, and expanding gives:

$$x^2+x+1 = x^2+a^2+2ax+b = x^2+(2a)x+(a^2+b)$$

The coefficients of x^2 are equal, the coefficients of x^1 should be equal, and the coefficients of x^0 should be equal. This gives the system of equations:

$$\begin{cases} 1 = 2a \\ 1 = a^2 + b \end{cases} \quad \text{which has solution} \quad \begin{cases} a = \frac{1}{2}, \\ b = \frac{3}{4}. \end{cases}$$

We can then rewrite:

$$\frac{1}{x^2+x+1} = \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

which we can now integrate:

$$\int \frac{1}{x^2+x+1} \cdot dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} dx \underset{\substack{u = x+\frac{1}{2} \\ du = dx}}{=} \int \frac{du}{u^2+\frac{3}{4}} = \int \frac{du}{\frac{3}{4}\cdot\left(\frac{4}{3}u^2+1\right)} = \frac{4}{3}\cdot\int \frac{du}{\left(\frac{2u}{\sqrt{3}}\right)^2+1} \underset{\substack{v = \frac{2u}{\sqrt{3}} \\ dv = \frac{2}{\sqrt{3}} du}}{=}$$

$$= \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\cdot dv}{v^2+1} = \frac{2}{\sqrt{3}}\int \frac{dv}{v^2+1} = \frac{2}{\sqrt{3}}\cdot \arctan(v) = \frac{2}{\sqrt{3}}\cdot \arctan\left(\frac{2u}{\sqrt{3}}\right) =$$

$$= \frac{2}{\sqrt{3}}\cdot \arctan\left(\frac{2x+1}{\sqrt{3}}\right).$$

Now:

$$\int_2^R \frac{1}{x^2+x+1}\cdot dx = \frac{2}{\sqrt{3}}\cdot \arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Bigg|_2^R = \frac{2}{\sqrt{3}}\cdot\left(\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \arctan\left(\frac{2\cdot 2+1}{\sqrt{3}}\right)\right) =$$

$$= \frac{2}{\sqrt{3}}\cdot\left(\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \arctan\left(\frac{5}{\sqrt{3}}\right)\right).$$

This gives:

$$\int_2^R \frac{1-x}{x^2+x+1} dx = \frac{-1}{2}\cdot\int_2^R \frac{2x+1}{x^2+x+1} dx + \frac{3}{2}\cdot\int_2^R \frac{1}{x^2+x+1}\cdot dx =$$

$$= \frac{-1}{2}\cdot\left(\ln|R^2+R+1| - \ln|7|\right) + \frac{3}{2}\cdot\left(\frac{2}{\sqrt{3}}\cdot\left(\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \arctan\left(\frac{5}{\sqrt{3}}\right)\right)\right) =$$

$$= \frac{1}{2}\cdot\ln|7| - \frac{1}{2}\cdot\ln|R^2+R+1| + \sqrt{3}\cdot\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \sqrt{3}\cdot\arctan\left(\frac{5}{\sqrt{3}}\right).$$

Finally, let's put all of these integrals together:

$$\int_2^\infty \frac{x}{x^3-1}\,dx = \int_2^\infty \frac{1}{3}\cdot\left(\frac{1}{x-1} + \frac{1-x}{x^2+x+1}\right)dx = \lim_{R\to\infty}\left(\frac{1}{3}\int_2^R \frac{dx}{x-1} + \frac{1}{3}\int_2^R \frac{1-x}{x^2+x+1}\,dx\right) =$$

$$= \lim_{R\to\infty}\left(\frac{1}{3}\cdot\left(\ln|R-1|\right) + \frac{1}{3}\cdot\left(\frac{1}{2}\cdot\ln|7| - \frac{1}{2}\cdot\ln|R^2+R+1| + \sqrt{3}\cdot\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \sqrt{3}\cdot\arctan\left(\frac{5}{\sqrt{3}}\right)\right)\right) =$$

$$= \lim_{R\to\infty}\left(\frac{\ln|R-1|}{3} + \frac{\ln|7|}{6} - \frac{\ln|R^2+R+1|}{6} + \frac{1}{\sqrt{3}}\arctan\left(\frac{2R+1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right)\right) =$$

$$= \lim_{R\to\infty}\left(\frac{\ln|7|}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{2\cdot\ln|R-1| - \ln|R^2+R+1|}{6} + \frac{1}{\sqrt{3}}\arctan\left(\frac{2R+1}{\sqrt{3}}\right)\right) =$$

$$= \lim_{R\to\infty}\left(\frac{\ln|7|}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{\ln|(R-1)^2| - \ln|R^2+R+1|}{6} + \frac{1}{\sqrt{3}}\arctan\left(\frac{2R+1}{\sqrt{3}}\right)\right) =$$

$$= \lim_{R\to\infty}\left(\frac{\ln|7|}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{1}{6}\cdot\ln\left|\frac{(R-1)^2}{R^2+R+1}\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2R+1}{\sqrt{3}}\right)\right) =$$

$$= \lim_{R\to\infty}\left(\frac{\ln|7|}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{1}{6}\cdot\ln\left|\frac{R^2-2R+1}{R^2+R+1}\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2R+1}{\sqrt{3}}\right)\right) =$$

$$= \frac{\ln|7|}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right) + \frac{1}{6}\cdot 0 + \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} =$$

$$= \frac{\pi}{2\cdot\sqrt{3}} + \frac{\ln(7)}{6} - \frac{1}{\sqrt{3}}\arctan\left(\frac{5}{\sqrt{3}}\right).$$