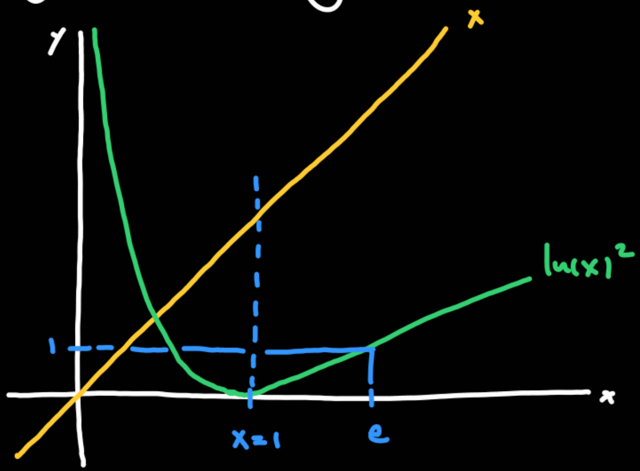
Problem 11.4.32: Determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^2}$.

Note that: $x > \ln(x)^2$ for $x \geq e$. We can see this in at least two ways.

(1) Drawing x and $\ln(x)^2$ gives:



So it is not true that $x > \ln(x)^2$ for all x, but it is true that $x > \ln(x)^2$ for $x \geq e$ (and even earlier, but $x = e$ will make computations easier).

(2) Consider the function $f(x) = x - \ln(x)^2$. We now show that $f(x) > 0$ for $x \geq e$. Since $f(e) = e - \ln(e)^2 = e - 1 > 0$, it suffices to show that $f(x)$ is increasing. We see this using derivatives: $f'(x) = 1 - 2 \cdot \ln(x) \cdot \frac{1}{x} = \frac{x - 2\cdot\ln(x)}{x}$. We claim that $f'(x) > 0$ for $x \geq e$. For this, it suffices to show that the numerator $x - 2\cdot\ln(x) > 0$.

Consider the function $g(x) = x - 2\cdot\ln(x)$, since $g(e) = e - 2\cdot\ln(e) = e - 2 > 0$, to see that $g(x) > 0$ for $x \geq e$ it suffices to show that $g(x)$ is increasing.

Again, we use derivatives for this: $g'(x) = 1 - 2\cdot\frac{1}{x} = \frac{x-2}{x}$, where it is now clear that $g'(x) > 0$ for $x \geq e$.

That is, $g'(x) = \frac{x-2}{x} > 0$ for $x \geq e$, so $g(x) = x - 2\cdot\ln(x)$ is increasing for $x \geq e$. Since $g(e) > 0$ then $g(x) > 0$ for $x \geq e$, so $f'(x) = \frac{x - 2\cdot\ln(x)}{x} > 0$ for $x \geq e$, so $f(x) = x - \ln(x)^2$ is increasing for $x \geq e$. Since $f(e) > 0$ then $f(x) > 0$ for $x \geq e$.

Thus $x - \ln(x)^2 = f(x) > 0$ so $x > \ln(x)^2$ for $x \geq e$.

Once we establish $x > \ln(x)^2$ for $x \geq e$, this means $n > \ln(n)^2$ for $n \geq 3$. Thus $\frac{1}{n} < \frac{1}{\ln(n)^2}$ for $n \geq 3$. Moreover, note that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges by the Integral Test. We now use the Comparison Test with $M = 3$, since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges then $\sum_{n=3}^{\infty} \frac{1}{\ln(n)^2}$ diverges. Finally, $\sum_{n=3}^{\infty} \frac{1}{\ln(n)^2} < \sum_{n=2}^{\infty} \frac{1}{\ln(n)^2}$ so $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^2}$ diverges.