Math 33A
Linear Algebra and Applications

Discussion for March 7-11, 2022

Problem 1.

Consider an $n \times m$ matrix A with $\text{rank}(A) = m$, and a singular value decomposition $A = U\Sigma V^T$. Show that the least-squares solution of a linear system $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}$$

Problem 2.

Consider the 4×2 matrix

$$A = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Find the least-squares solution of the linear system

$$A\vec{x} = \vec{b} \quad \text{where} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Problem 3.

(a) Explain how any square matrix A can be written as $A = QS$, where Q is orthogonal and S is symmetric positive semidefinite. This is called the polar decomposition of A.

(b) Is it possible to write $A = S_1Q_1$, where Q_1 is orthogonal and $S1$ is symmetric positive semidefinite?

Problem 4.

Find a polar decomposition $A = QS$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

Draw a sketch showing $S(C)$ and $A(C) = Q(S(C))$, where C is the unit circle centered at the origin.

Problem 5.

Show that a singular value decomposition $A = U\Sigma V^T$ can be written as

$$A = \sigma_1\vec{u_1}\vec{v_1}^T + \cdots + \sigma_r\vec{u_r}\vec{v_r}^T.$$

Problem 6.

Find a decomposition $A = \sigma_1\vec{u_1}\vec{v_1}^T + \sigma_2\vec{u_2}\vec{v_2}^T$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$