Math 33A
Linear Algebra and Applications

Final

Instructions: You have 24 hours to complete this exam. There are 14 questions, worth a total of 100 points. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ____________________________________

ID number: _________________________________

Section: ___________________________________

Question	Points	Score
1	5	
2	5	
3	7	
4	8	
5	8	
6	7	
7	8	
8	8	
9	8	
10	8	
11	7	
12	7	
13	7	
14	7	
Total:	100	

Problem 1. *5pts.*

Determine whether the following statements are true or false.

(a) If A and B are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

(b) The span of vectors $\vec{v_1}, \ldots, \vec{v_n}$ consists of all linear combinations of vectors $\vec{v_1}, \ldots, \vec{v_n}$.

(c) If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

(d) If A and B are symmetric $n \times n$ matrices, then AB must be symmetric as well.

(e) There exists a nonzero 4×4 matrix A such that $\det(A) = \det(4A)$.

Problem 2. *5pts.*

Determine whether the following statements are true or false.

(a) There exist invertible 2×2 matrices A and B such that $\det(A+B) = \det(A)+\det(B)$.

(b) The trace of any square matrix is the sum of its diagonal entries.

(c) If vector $\vec{v}$ is an eigenvector of both A and B, then $\vec{v}$ is an eigenvector of AB.

(d) If matrix A is positive definite, then all the eigenvalues of A must be positive.

(e) The function $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 5x_2$ is a quadratic form.

Problem 3. *7pts.*

Consider a linear system of four equations with three unknowns. We are told that the system has a unique solution. What does the reduced row-echelon form of the coefficient matrix of this system look like? Explain your answer.

Solution: There are three leading variables, so there must be a leading one in each column, so the reduced-row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 4. *8pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by reflection about the $x - z$-plane.

Solution: The matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 5. *8pts.*

Find the inverse of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 22 \\ -16 \\ 8 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 13 \\ -3 \\ 9 \\ 4 \end{bmatrix} + x_3\begin{bmatrix} 8 \\ -2 \\ 7 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

Solution: The inverse is the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -2 \\ 4 \\ -9 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 5 \\ -9 \\ 17 \end{bmatrix} + x_3\begin{bmatrix} 9 \\ -22 \\ 41 \\ -80 \end{bmatrix} + x_4\begin{bmatrix} -25 \\ 60 \\ -112 \\ 222 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

Problem 6. *7pts.*

Consider the plane $2x_1 - 3x_2 + 4x_3 = 0$. Find a basis $\mathfrak{B}$ of this plane such that

$$\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution: There are multiple solutions, one is

$$\mathfrak{B} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -4/3 \\ -4/3 \\ -1 \end{bmatrix} \right\}.$$

Problem 7. *8pts.*

Find the matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} \quad \text{with respect to the basis} \quad \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solution: We have

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Problem 8. *8pts.*

Find the QR factorization of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Solution: We have

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

so

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}, \quad \text{and} \quad R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

is the QR factorization.

Problem 9. *8pts.*

Find all the least-squares solutions $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of A, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

Solution: The solutions are

$$\vec{x}^* = \begin{bmatrix} 1 - 3t \\ t \end{bmatrix}$$

where t is an arbitrary real constant.

Problem 10. *8pts.*

Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}.$$

Solution: We have

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix} = 288.$$

Problem 11. *7pts.*

Use Cramer's rule to solve the system

$$\begin{aligned} 2x + 3y &= 8 \\ 4y + 5z &= 3 \\ 6x + 7z &= -1. \end{aligned}$$

Solution: The solutions are

$$x = \frac{\det \begin{bmatrix} 8 & 3 & 0 \\ 3 & 4 & 5 \\ -1 & 0 & 7 \end{bmatrix}}{146} = 1,$$

$$y = \frac{\det \begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 6 & -1 & 7 \end{bmatrix}}{146} = 2,$$

$$x = \frac{\det \begin{bmatrix} 2 & 3 & 8 \\ 0 & 4 & 3 \\ 6 & 0 & -1 \end{bmatrix}}{146} = -1.$$

Problem 12. *7pts.*

Show that 4 is an eigenvalue of

$$\begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix}$$

and find all corresponding eigenvectors.

Solution: Solving

$$\begin{bmatrix} -6 & 6 \\ -15 & 13 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 4 \begin{bmatrix} a \\ b \end{bmatrix}$$

we find

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{3t}{5} \\ t \end{bmatrix}$$

where t is an arbitrary non-zero real constant.

Problem 13. *7pts.*

Suppose a real 3×3 matrix A has only two distinct eigenvalues. Suppose that $\mathrm{tr}(A) = 1$ and $\det(A) = 3$. Find the eigenvalues of A with their algebraic multiplicities.

Solution: We have $1 = \mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$ and $\det(A) = \lambda_1\lambda_2\lambda_3$ with $\lambda_1 = \lambda_2 \neq \lambda_3$ so $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 3$.

Problem 14. *7pts.*

Find the matrix of the quadratic form $q(x_1, x_2) = 6x_1^2 - 7x_1x_2 + 8x_2^2$.

Solution: The matrix is

$$\begin{bmatrix} 6 & -7/2 \\ -7/2 & 8 \end{bmatrix}.$$