Math 33A
Linear Algebra and Applications

Midterm 1

Instructions: You have 24 hours to complete this exam. There are 7 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________
ID number: ______________________________
Section: ______________________________

Question	Points	Score
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total:	100	

Problem 1. *10pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) If matrix A is in reduced row-echelon form, then at least one of the entries in each column must be 1.

F

(b) Consider an $n \times m$ matrix A. Can you transform $\mathrm{rref}(A)$ into A by a sequence of elementary row operations?

T

(c) The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

T

(d) If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

F

(e) If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

T

Problem 2. *15pts.*

Find all solutions of the linear system

$$\begin{aligned} x + 2y + 3z &= a \\ x + 3y + 8z &= b \\ x + 2y + 2z &= c \end{aligned}$$

where a, b, c are arbitrary constants.

Solution: The solutions are $x = 10a - 2b - 7c$, $y = -6a + b + 5c$, $z = a - c$.

Problem 3. *15pts.*

Determine whether the vector

$$\begin{bmatrix} 1 \\ -1 \\ 8 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 10 \\ 1 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 9 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$

Solution: Never, these vectors are linearly independent.

Problem 4. *15pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by a rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

Solution: The matrix is

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}.$$

Problem 5. *15pts.*

Decide whether the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

is invertible. If it is, find the inverse.

Solution: The matrix is invertible, with inverse

$$\begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}.$$

Problem 6. *15pts.*

Describe, geometrically and algebraically, the image and kernel of the transformation in $\mathbb{R}^3$ given by the orthogonal projection onto the plane $x + 2y + 3z = 0$. In particular, find a basis of the image and a basis of the kernel.

Solution: The linear transformation has image $\mathrm{im}(T) = \{(x, y, z) \in \mathbb{R}^3 | x+2y+3z = 0\}$, namely the whole plane $x+2y+3z = 0$, with basis $\{[-2\,1\,0], [-3\,0\,1]\}$, and kernel $\ker(T) = \mathrm{span}([1\,2\,3])$, namely the line spanned by the vector $\vec{v} = [1\,2\,3]$, with basis $\{[1\,2\,3]\}$.

Problem 7. *15pts.*

Find a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix}1\\2\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}3\\5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}3\\4\end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Solution: We are looking for a matrix S such that $S\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$ and $S\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$. This has solution $S = \begin{bmatrix}12 & -7\\14 & -8\end{bmatrix}$, so the desired basis is $\left\{\begin{bmatrix}12\\14\end{bmatrix}, \begin{bmatrix}-7\\-8\end{bmatrix}\right\}$.