**Math 33A**
**Linear Algebra and Applications**

**Midterm 2**

**Instructions:** You have 24 hours to complete this exam. There are 7 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________
ID number: ___________________________
Section: _____________________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 15 | |
| Total: | 100 | |

**Problem 1**. *10pts.*

Determine whether the following statements are true or false.

(a) If $A$ and $B$ are symmetric $n \times n$ matrices, then $ABBA$ must be symmetric as well.

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(b) If $A$ is an invertible matrix such that $A^{-1} = A$, then $A$ must be orthogonal.

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(c) If $V$ is a subspace of $\mathbb{R}^n$ and $\vec{x}$ is a vector in $\mathbb{R}^n$, then the inequality $\vec{x} \cdot (\text{proj}_V \vec{x}) \geq 0$ must hold.

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(d) If matrix $B$ is obtained by swapping two rows of an $n \times n$ matrix $A$, then the equation $\det(B) = -\det(A)$ must hold.

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(e) There exist real invertible $3 \times 3$ matrices $A$ and $S$ such that $S^T A S = -A$.

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**Problem 2**. *15pts.*

Consider the vectors

$$\vec{u_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u_2} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u_3} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

in $\mathbb{R}^4$. Is there a vector $\vec{u_4}$ in $\mathbb{R}^4$ such that $\mathfrak{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is an orthonormal basis? If so, how many such vectors are there?

**Solution:** Yes, there are two possible vectors

$$\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

which make $\mathfrak{B}$ into an orthonormal basis.

**Problem 3**. *15pts.*

Find the $QR$ factorization of the following matrix.

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix}.$$

**Solution:** Since the first matrix has orthonormal columns and the second matrix is upper triangular, we are almost given the $QR$ factorization of $M$, we only need to fix the $-4$. When we multiply these two matrices, the second column of the first matrix will be multiplied by the second row of the second matrix. If we change the signs of both this columns, the product will be the same, and now we have

$$M = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}$$

which is the $QR$ factorization of $M$ with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}.$$

Algebraically, the above explanation is encoded as

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix} \right) =$$

$$= \frac{1}{2}\left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}.$$

**Problem 4**. *15pts.*

Find an orthogonal matrix of the form

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}.$$

**Solution:** There are two

$$\begin{bmatrix} 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & 1/\sqrt{18} \\ 1/3 & 0 & -4/\sqrt{18} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \end{bmatrix}.$$

**Problem 5**. *15pts.*

Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of $A$, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

**Solution:** The least-squares solution is

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and $\vec{b} - A\vec{x}^* = \vec{0}$, so $\vec{x}^*$ is the exact solution. In particular, $\vec{b}$ is inside the plane $\text{im}(A)$, and it coincides exactly with $A\vec{x}^*$.

**Problem 6**. *15pts.*

Consider a $4 \times 4$ matrix $A$ with rows $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. If $\det(A) = 8$, find

$$\det \begin{bmatrix} 6\vec{v_1} + 2\vec{v_4} \\ \vec{v_2} \\ \vec{v_3} \\ 3\vec{v_1} + \vec{v_4} \end{bmatrix}.$$

**Solution:** The first and the last rows are not linearly independent since we can obtain the last row as 1/2 of the first row, so the determinant is zero.

**Problem 7**. *15pts.*

Find the classical adjoint of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

and use the result to find $A^{-1}$.

**Solution:** We can find

$$\operatorname{adj}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and $\det(A) = -1$, and thus

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$