Math 33A
Linear Algebra and Applications

Final

Instructions: You have 24 hours to complete this exam. There are 14 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________

ID number: ______________________________

Section: ______________________________

Question	Points	Score
1	5	
2	5	
3	7	
4	8	
5	8	
6	7	
7	7	
8	8	
9	7	
10	8	
11	8	
12	8	
13	7	
14	7	
Total:	100	

Problem 1. *5pts.*

Determine whether the following statements are true or false.

(a) If matrix A is in reduced row-echelon form, then at least one of the entries in each column must be 1.

F

(b) If A and B are any two 3×3 matrices of rank 2, then A can be transformed into B by means of elementary row operations.

F

(c) If A and B are matrices of the same size, then the formula $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$ must hold.

F

(d) There exists an invertible $n \times n$ matrix with two identical rows.

F

(e) The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

F

Problem 2. *5pts.*

Determine whether the following statements are true or false.

(a) If a square matrix A is invertible, then its classical adjoint $\mathrm{adj}(A)$ is invertible as well.

T

(b) All invertible matrices are diagonalizable.

F

(c) If two matrices A and B have the same characteristic polynomials, then they must be similar.

F

(d) All symmetric matrices are diagonalizable.

T

(e) If A is an invertible symmetric matrix, then A^2 must be positive definite.

T

Problem 3. *7pts.*

If the rank of a 5×3 matrix A is 3, what is $\text{rref}(A)$?

Solution: We must have a leading one in each column.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 4. *8pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

Solution: The matrix is

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 5. *8pts.*

Find the inverse of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 5 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

Solution: The inverse is the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 5 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

from $\mathbb{R}^4$ to $\mathbb{R}^4$.

Problem 6. *7pts.*

For which value(s) of the constant k do the vectors below form a basis of $\mathbb{R}^4$?

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ k \end{bmatrix}.$$

Solution: They form a basis whenever $k \neq 29$.

Problem 7. *7pts.*

Consider the plane $x_1 + 2x_2 + x_3 = 0$. Find a basis $\mathfrak{B}$ of this plane such that

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Solution: There are multiple solutions, one is

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Problem 8. *8pts.*

Find the matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix} \vec{x} \quad \text{with respect to the basis} \quad \vec{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$$

Solution: We have

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 9. *7pts.*

Find an orthonormal basis of the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Solution: Since

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

a basis of $\ker(A)$ is

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and after applying the Gram-Schmidt process we obtain the orthonormal basis

$$\mathfrak{R} = \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} \right\}.$$

Problem 10. *8pts.*

Find all the least-squares solutions $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of A, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

Solution: The solutions are

$$\vec{x}^* = \begin{bmatrix} t - \frac{7}{6} \\ 1 - 2t \\ t \end{bmatrix}.$$

Problem 11. *8pts.*

Use Cramer's rule to solve the system

$$\begin{aligned} 3x + 5y + 3z &= 25 \\ 7x + 9y + 19z &= 65 \\ -4x + 5y + 11z &= 5. \end{aligned}$$

Solution: The solution is $x = 4$, $y = 2$, $z = 1$.

Problem 12. *8pts.*

For the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

find all (real) eigenvalues. Then find a basis of each eigenspace, and diagonalize the matrix, if it can be done.

Solution: The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, an eigenbasis is

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

and we can diagonalize it as

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

Problem 13. *7pts.*

Consider a matrix of the form

$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$ are positive. Suppose the matrix A has three distinct real eigenvalues. What can you say about the signs of the eigenvalues? Is the eigenvalue with the largest absolute value positive or negative?

Solution: Two eigenvalues must be negative, one must be positive, and the one with the largest absolute value must be positive.

Problem 14. *7pts.*

Find the matrix of the quadratic form $q(x_1, x_2) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 6x_1x_3 + 7x_2x_3$.

Solution: The matrix is
$$\begin{bmatrix} 3 & 0 & 3 \\ 0 & 4 & 7/2 \\ 3 & 7/2 & 5 \end{bmatrix}.$$