Math 33A
Linear Algebra and Applications

Practice Midterm 1

Instructions: You have 24 hours to complete this exam. There are 7 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ______________________________
ID number: ______________________________
Section: ______________________________

Question	Points	Score
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total:	100	

Problem 1. *10pts.*

Determine whether the following statements are true or false.

(a) If the system $A\vec{x} = \vec{b}$ has a unique solution, then A must be a square matrix.

F

(b) The system $A\vec{x} = \vec{b}$ is inconsistent if and only if $\mathrm{rref}(A)$ contains a row of zeros.

F

(c) If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.

T

(d) If matrix A commutes with matrix B, and B commutes with matrix C, then A must commute with C.

F

(e) If vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are linearly independent, then vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

T

Problem 2. *15pts.*

Consider the linear system

$$
\begin{aligned}
x + y - z &= 2\\
x + 2y + z &= 3\\
x + y + (k^2 - 5)z &= k
\end{aligned}
$$

where k is an arbitrary constant. For which values of k does this system have a unique solution? For which value(s) of k does the system have infinitely many solutions? For which value(s) of k is the system inconsistent?

Solution: This system can be reduced to

$$
\begin{aligned}
x - 3z &= 1\\
y + 2z &= 1\\
(k^2 - 4)z &= k - 2
\end{aligned}
$$

so the system has a unique solution if $k \neq \pm 2$. If $k = 2$ the system has infinitely many solutions. If $k = -2$ the system has no solutions.

Problem 3. *15pts.*

Determine the values of the constants a, b, c, d for which the vector

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 0 \\ 5 \\ 6 \end{bmatrix}.$$

Solution: Consider the matrix

$$\begin{bmatrix} a & 0 & 1 & 2 \\ b & 0 & 0 & 0 \\ c & 3 & 4 & 5 \\ d & 0 & 0 & 6 \end{bmatrix}.$$

By permuting the first and second row, we see that the only way in which we can reduce this matrix to the identity is if $b \neq 0$. This would mean that the vectors are linearly independent. Thus to have linear dependence the constants a, c, d can take any value, and $b = 0$.

Problem 4. *15pts.*

Find the matrix of the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the reflection about the plane $y = z$.

Solution: Computing the reflection for each of the unitary vectors in each of the three directions, we find that $T(\vec{e_1}) = e_1$, $T(\vec{e_2}) = e_3$, and $T(\vec{e_3}) = e_2$. Thus the matrix is

$$\begin{bmatrix} | & | & | \\ T(\vec{e_1}) & T(\vec{e_2}) & T(\vec{e_3}) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Problem 5. *15pts.*

Decide whether the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}$$

is invertible. If it is, find the inverse.

Solution: The matrix is invertible, with inverse

$$\begin{bmatrix} -6 & 9 & -5 & 1 \\ 9 & -1 & -5 & 2 \\ -5 & -5 & 9 & -3 \\ 1 & 2 & -3 & 1 \end{bmatrix}.$$

Problem 6. *15pts.*

Let $\vec{v}$ be a vector in $\mathbb{R}^3$. Describe, geometrically and algebraically, the image and kernel of the transformation T from $\mathbb{R}^3$ to $\mathbb{R}$ given by taking the dot product with $\vec{v}$. In particular, find a basis of the image and a basis of the kernel.

Solution: The linear transformation is $T(\vec{x}) = \vec{v} \cdot \vec{x}$.

1. If $\vec{v} = \vec{0}$ then $T(\vec{x}) = \vec{0}$ for all $\vec{x}$ in $\mathbb{R}^3$, so $\text{im}(T) = \{0\}$ with basis $\{\}$ the set with no elements, and $\ker(T) = \mathbb{R}^3$ with basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$.
2. If $\vec{v} \neq \vec{0}$ then $\text{im}(T) = \mathbb{R}$ with basis $\{[1]\}$, and $\ker(T) = \{\vec{x} \in \mathbb{R}^3 | \vec{v} \cdot \vec{x} = 0\}$ is the plane with normal vector $\vec{v}$, that is, the plane given by the equation $v_1 x + v_2 y + v_3 z = 0$ where v_1, v_2, v_3 are the components of $\vec{v}$.
   (a) If $v_1 = v_2 = 0$ then $\ker(T)$ has basis $\{[1\,0\,0], [0\,1\,0]\}$. Similarly if any two other pair v_1, v_3 or v_2, v_3 are both zero, a similar basis follows.
   (b) If $v_1 = 0$ then $\ker(T)$ has basis $\{[0\,1\, -v_2/v_3], [1\,1\, -v_2/v_3]\}$. Similarly if any other v_2 or v_3 is zero, a similar basis follows.
   (c) If v_1, v_2, v_3 are all non-zero then $\ker(T)$ has basis $\{[-v_2/v_1\,1\,0], [-v_3/v_1\,0\,1]\}$.

Problem 7. *15pts.*

Consider the basis of $\mathbb{R}^2$ given by $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $\mathfrak{R} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$. Find a matrix P such that $\left[\vec{x}\right]_{\mathfrak{R}} = P \left[\vec{x}\right]_{\mathfrak{B}}$.

Solution: Since $\vec{x} = S_{\mathfrak{R}} \left[\vec{x}\right]_{\mathfrak{R}}$ and $\vec{x} = S_{\mathfrak{B}} \left[\vec{x}\right]_{\mathfrak{B}}$, with $S_{\mathfrak{R}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $S_{\mathfrak{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, solving the equation $S_{\mathfrak{R}} \left[\vec{x}\right]_{\mathfrak{R}} = S_{\mathfrak{B}} \left[\vec{x}\right]_{\mathfrak{B}}$ gives $P = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$.