Example: Is the zero matrix in rref? Yes!

Example:

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ No solutions.

(b) $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

$x + 2y = 1 \rightsquigarrow x = 1 - 2y$

$z = 1$ $\quad y$ is any real number

Infinite solutions.

(c) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ One solution.

A system is called <u>consistent</u> if it has at least one solution.

If a system has <u>no</u> solutions, we call it <u>inconsistent</u>.

<u>Theorem:</u> A linear system is inconsistent if and only if its reduced row-echelon form has a row of the form $[0 \cdots 0 \mid 1]$. If a linear system is consistent then:

(a) it has infinitely many solutions if we have at least one free variable,

(b) it has one solution if all the variables are leading.

$\left[\begin{array}{c|c} 1 & 1 \\ 0 & 0 \end{array}\right]$ $\quad x = 1$

The <u>rank</u> of a matrix A is the number of leading 1's in its rref.

Example: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$ The rank is ?

Example: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} & & \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ The rank is 2.

Theorem: Consider a system of n equations and m variables. Then:

(i) We have $\operatorname{rank}(A) \le n$ and $\operatorname{rank}(A) \le m$.

(ii) If $\operatorname{rank}(A) = n$, then the system is consistent.

(iii) If $\operatorname{rank}(A) = m$, then the system has at most one solution.

(iv) If $\operatorname{rank}(A) < m$, then the system has zero or infinitely many solutions.

Why ?

Example:

1. Suppose we have a system with fewer equations than variables.

   How many solutions would it have? Zero or infinitely many.

2. Suppose we have a system with n equations and n variables.

   When do we have exactly one solution? $\operatorname{Rank}(A) = n$.

Addition: $C = A + B$ $\qquad c_{ij} = a_{ij} + b_{ij}$

Scalar multiplication: $C = kA$ $\qquad c_{ij} = k\,a_{ij}$

Dot product: $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$ $\qquad [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Multiplication $A\vec{x}$:

$\mathbb{R}^m$

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

$\mathbb{R}^m$

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m$$

$\mathbb{R}^n$

Algebraic rules:

$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ $\qquad$ $A(k\vec{x}) = kA\vec{x}$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$$