$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \cdot \vec{v}_m$$

$$A\vec{x} = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vdots \\ \vec{w}_n \cdot \vec{x} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 12 \\ 24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} [1\ 2] \cdot \left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right] \\ [3\ 4] \cdot \left[\begin{smallmatrix} 5 \\ 6 \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

A vector $\vec{v}$ is a <u>linear combination</u> of the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in $\mathbb{R}^n$ if there are scalars $a_1, \ldots, a_m$ such that $\vec{v} = a_1 \vec{v}_1 + \cdots + a_m \vec{v}_m$.

Given a system of linear equations $[A \mid \vec{b}]$ we can write this as an equality of matrices: $A\vec{x} = \vec{b}$ where $\vec{x}$ is the vector of variables.

<u>Example:</u>

$$\begin{aligned} 2x + 8y + 4z &= 2 \\ 2x + 5y + z &= 5 \\ 4x + 10y - z &= 1 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

<u>Example:</u>

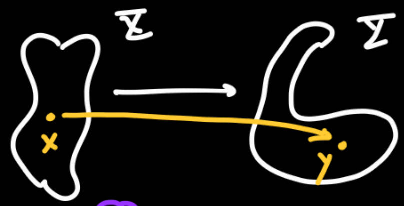
1. There exists a 3×4 matrix of rank 4. False!

2. There exists a system of 3 eqs, 3 unks, with 3 sols. False!

3. If A is a 3×4 matrix of rank 3, then the system $A\vec{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ must have infinitely many solutions. True!

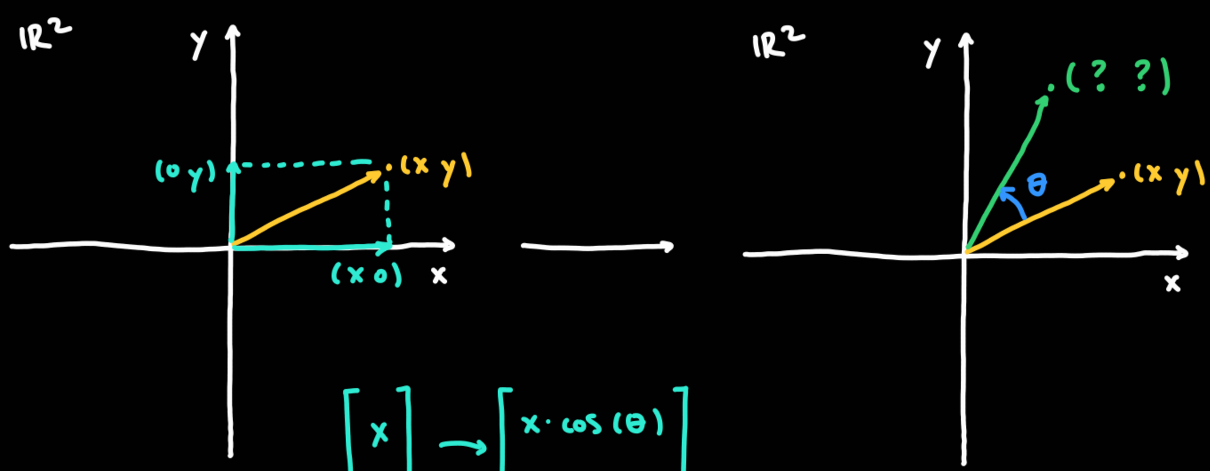
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & 1 \\ 0 & 1 & 0 & * & 2 \\ 0 & 0 & 1 & \underbrace{*}_{\text{free variable}} & 3 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & * & 1 \\ 0 & 1 & 0 & * & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

A **function** T is a rule that assigns to elements x in its domain X unique elements y in its range Y. $T(x) = y$. T:



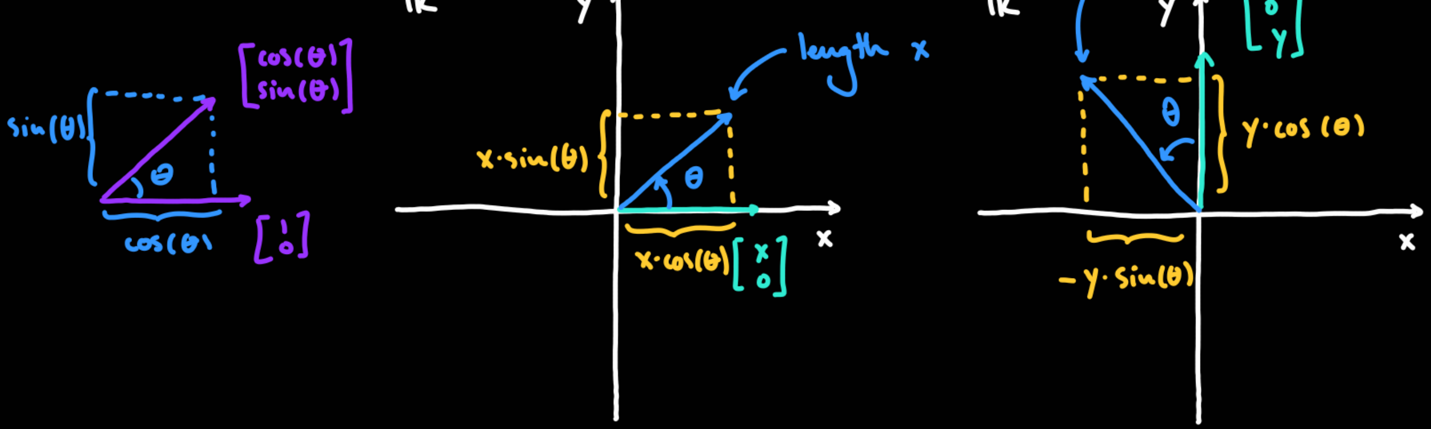
A **linear transformation** is a function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that there is an $n \times m$ matrix A with $T(\vec{x}) = A\vec{x}$ with $\vec{x}$ in $\mathbb{R}^m$.

Example: Consider the function from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by a rotation of angle θ.



$$\begin{bmatrix} x \\ 0 \end{bmatrix} \to \begin{bmatrix} x\cdot\cos(\theta) \\ x\cdot\sin(\theta) \end{bmatrix}$$

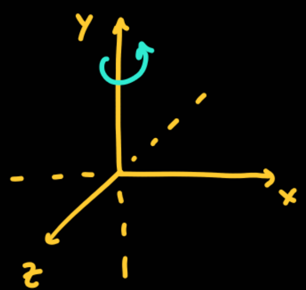
$$\begin{bmatrix} 0 \\ y \end{bmatrix} \to \begin{bmatrix} y\cdot(-\sin(\theta)) \\ y\cdot\cos(\theta) \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x\cdot\cos(\theta) - y\cdot\sin(\theta) \\ x\cdot\sin(\theta) + y\cdot\cos(\theta) \end{bmatrix}$$



This is a linear transformation! The matrix associated to it is:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cdot \cos(\theta) - y \cdot \sin(\theta) \\ x \cdot \sin(\theta) + y \cos(\theta) \end{bmatrix}$$



Theorem: Let T be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then the matrix associated to T is: $\begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_m) \\ | & & | \end{bmatrix}$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Example:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \qquad \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Theorem: A function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a linear transformation i.f.o.i.f.:

(i) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$.

(ii) $T(\lambda \vec{v}) = \lambda T(\vec{v})$. λ in $\mathbb{R}$