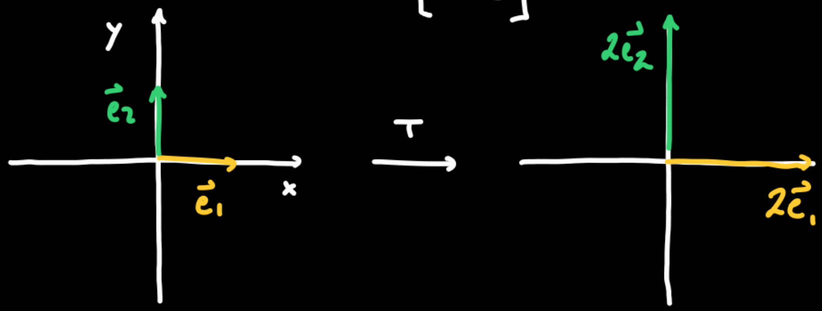
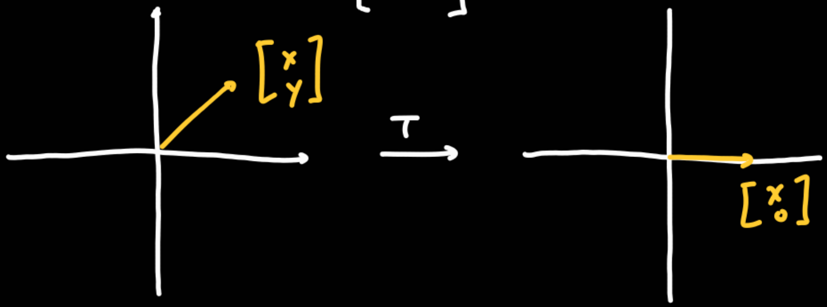
Example:

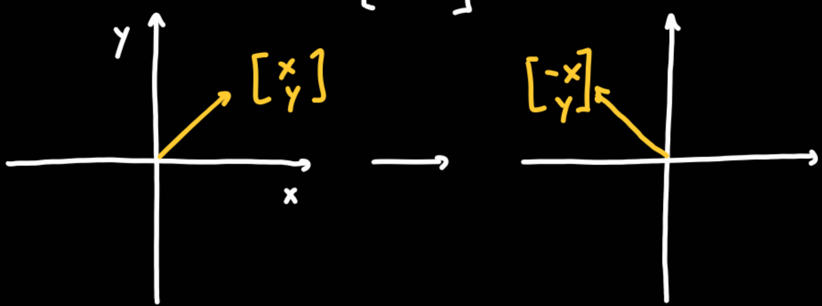
1. What does the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ do to $\mathbb{R}^2$?



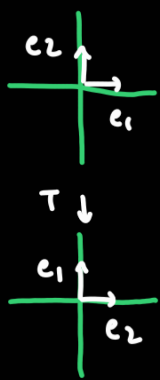
2. What does the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ do to $\mathbb{R}^2$?

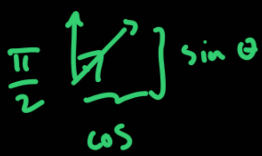


3. What does the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ do to $\mathbb{R}^2$?



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\theta = \frac{\pi}{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$





$\vec{e}_1 \to \vec{e}_2$

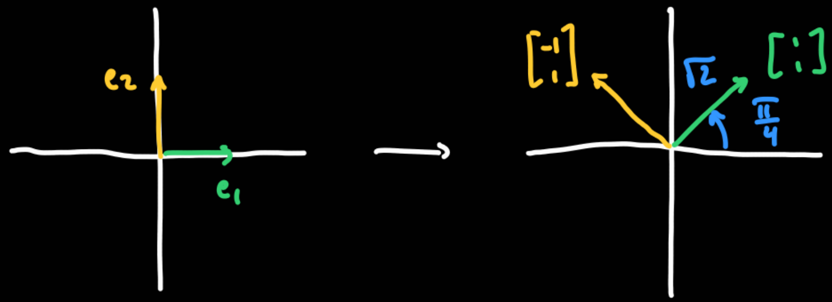
$\vec{e}_2 \to \vec{e}_1$

$\theta = \frac{-\pi}{2}$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4. What does the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ do to $\mathbb{R}^2$?

(i) rotation by $\frac{\pi}{4}$.

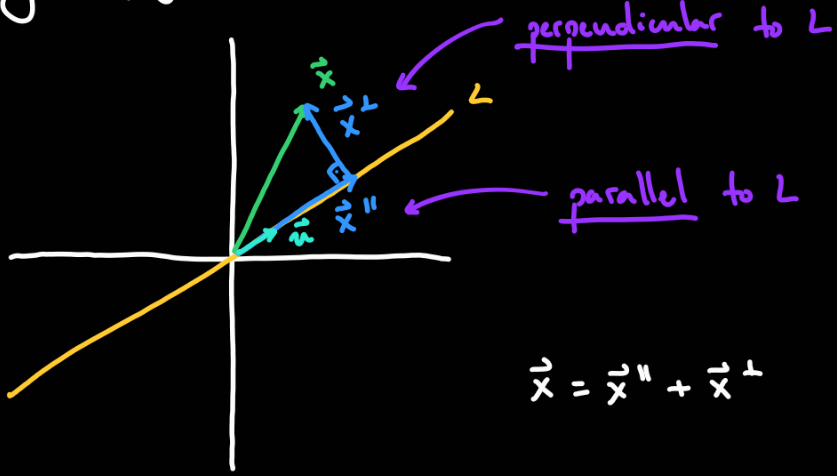
(ii) dilation / scaling / zoom / stretch by $\sqrt{2}$.



Scaling:

Is given by multiplying by a diagonal matrix: $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ k in $\mathbb{R}$.

Orthogonal projections:



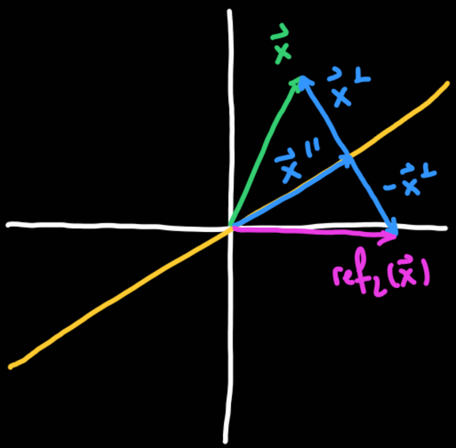
$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

$\mathrm{proj}_L(\vec{x}) = \vec{x}^{\parallel}$ is the <u>orthogonal projection</u> of $\vec{x}$ onto L.

Let $\vec{u}$ be unitary and parallel to L. The dot product $\vec{x}\cdot\vec{u}$ is exactly the length of $\vec{x}^{\parallel}$. Thus: $\vec{x}^{\parallel} = \mathrm{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u}$.

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ unitary $\quad \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$ ← matrix of projection on the line defined by $\vec{u}$.

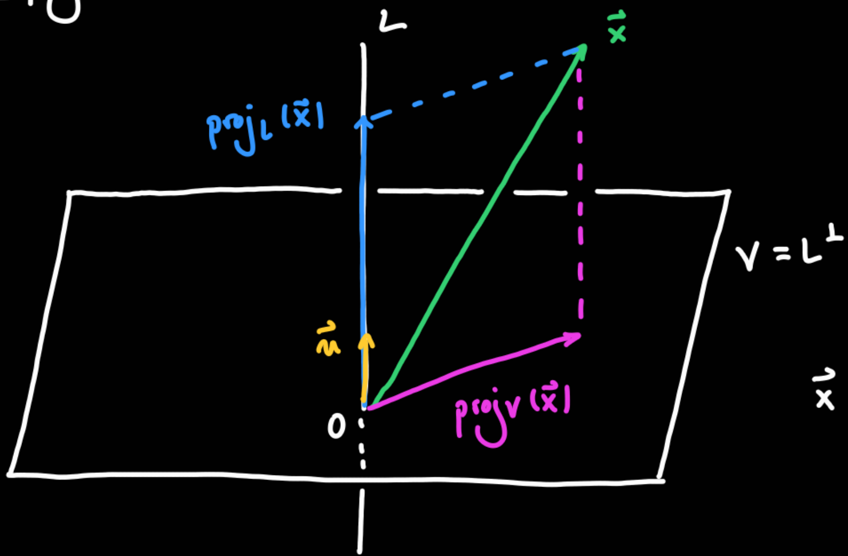
Reflection:



The *reflection* of $\vec{x}$ onto L is $\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - x^{\perp}$.

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ unitary $\qquad \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

Orthogonal projection in $\mathbb{R}^3$.



$\vec{x} = \text{proj}_V(\vec{x}) + \text{proj}_L(\vec{x})$

(i) $\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}$

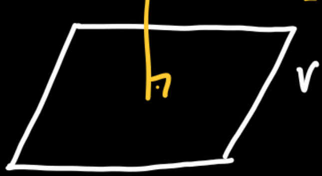
(ii) $\text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x})$

(iii) $\text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x})$

(iv) $\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x})$

Example:

$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$



$V: \quad 2x_1 + x_2 - 2x_3 = 0$

$\vec{x}: \quad \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} \qquad \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad v_n$$

$$5 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} + (-2) \cdot \frac{(-2)}{3}$$

(i) $\operatorname{proj}_L(\vec{x}) = \overbrace{(\vec{x} \cdot \vec{u})}\, \vec{u} = \begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix} \qquad 6 \cdot \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

(ii) $\operatorname{proj}_V(\vec{x}) = \vec{x} - \operatorname{proj}_L(\vec{x}) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(iii) $\operatorname{ref}_L(\vec{x}) = \operatorname{proj}_L(\vec{x}) - \operatorname{proj}_V(\vec{x}) = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$

(iv) $\operatorname{ref}_V(\vec{x}) = \operatorname{proj}_V(\vec{x}) - \operatorname{proj}_L(\vec{x}) = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$