<u>Recall:</u> We saw that we could "compose / concatenate" linear transformations. Specifically,

make this formal algebraically.

we saw a rotation with a scaling.

Let $T$ be a linear transformation given by $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, let $S$ be a linear transformation given by $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. We want to find $\vec{z} = T(S(\vec{x}))$. We do this in two steps:

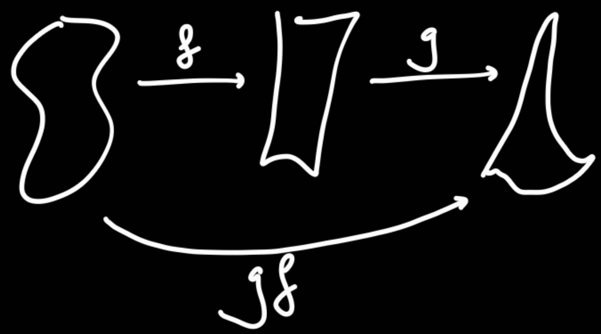
first $S$

then $T$

$\vec{y} = S(\vec{x})$ , then $\vec{z} = T(\vec{y})$. Looking at the equations given by these equalities:

$$\begin{aligned} y_1 &= x_1 + 2x_2 & \text{and} \quad z_1 &= 6y_1 + 7y_2 & \text{so} \quad z_1 &= 27x_1 + 47x_2 \\ y_2 &= 3x_1 + 5x_2 & z_2 &= 8y_1 + 9y_2 & z_2 &= 35x_1 + 61x_2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{S} \qquad \underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{T} \qquad \underbrace{\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{TS}$$




This computation should mean that $\vec{z} = TS(\vec{x})$ is given by $\begin{bmatrix} 27 & 47 \\ 35 & 61 \end{bmatrix}$, and this should be the product of the matrices giving $T$ and $S$, namely $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

<u>Matrix multiplication:</u>

column      row

Let $B$ be an $n \times p$ matrix, let $A$ be an $q \times m$. If (and only if) $p = q$ then the <u>product</u> $BA$ is the matrix associated to the linear transformation

$T(\vec{x}) = B(A\vec{x})$. This product $BA$ is an $n \times m$ matrix.

$$\begin{array}{ccccc} \vec{x} & & A\vec{x} & & \\ \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^p & & \\ & & \mathbb{R}^p & \xrightarrow{B} & \mathbb{R}^n \end{array}$$

<u>Theorem:</u> Let $B$ be an $n \times p$ matrix and $A$ a $p \times m$ matrix. Then:

(i) $$BA = B\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ B\vec{v}_1 & \cdots & B\vec{v}_m \\ | & & | \end{bmatrix}.$$

(ii) $$C = BA = \begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$$ has entries $\underline{c_{ij}} = \vec{w}_i \cdot \vec{v}_j = \sum_{k=1}^{p} b_{ik} a_{kj}$.

<u>Example:</u> Matrix multiplication is <u>not</u> commutative:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

<u>Algebraic rules:</u>

(i) If $A$ is an $n \times m$ matrix: $A \overbrace{I_m}^{\text{identity matrix } m \times m} = A = \underbrace{I_n}_{\text{identity matrix } n \times n} A$

(ii) Matrix multiplication is associative: $(AB)C = A(BC)$

(iii) Matrix multiplication distributes over addition:

$$(\underbrace{A+B}_{\text{1st}})C = \underbrace{(AC)}_{\text{1st}} + \underbrace{(BC)}_{\text{1st}} \quad \text{and} \quad A'(B'+C') = A'B' + A'C'$$

(iv) Multiplication by scalars can be factored out:

$$(kA)B = k(AB) = A(kB).$$