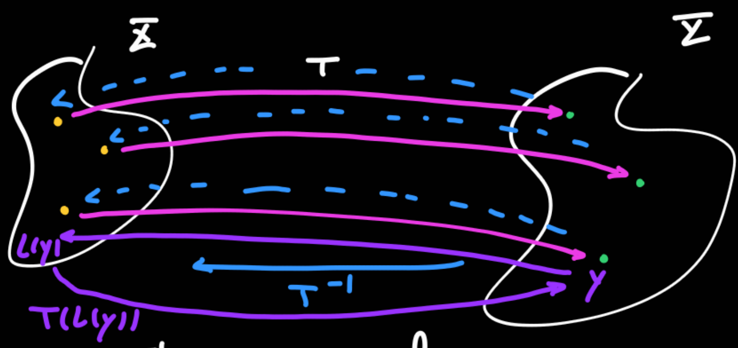
Recall: A function T is invertible if for each y in Y there is a unique x in X such that $T(x) = y$.



T^{-1} the inverse of T

A function T has inverse $L = T^{-1}$ if and only if:

$$T(L(y)) = y \text{ for all } y, \quad \text{and} \quad L(T(x)) = x \text{ for all } x.$$

Let A be a square matrix, we say that A is invertible if its associated linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible.

T^{-1} will be linear, and its associated matrix is denoted A^{-1}.

Theorem: Let A be an $n \times n$ matrix.

$n \times n$ identity matrix

(i) A is invertible if and only if $\operatorname{rank}(A) = n$, i.o.i.f. $\operatorname{rref}(A) = I_n$.

(ii) $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$ i.o.i.f. A is invertible.

Example: Let A be an $n \times n$ matrix. The equation $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If A is invertible this is the only solution. If A is not invertible there are infinitely many solutions invertible then we have infinitely many solutions.

Example: Rotation by an angle θ counterclockwise is invertible.

The inverse is rotation by angle θ clockwise.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A \qquad A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{matrix} -\theta & -\theta \\ -\theta & -\theta \end{matrix}$$

Theorem: To find the inverse of an $n \times n$ matrix A, compute $\mathrm{rref}([A \mid I_n])$.

(i) If $\mathrm{rref}([A \mid I_n]) = [I_n \mid B]$ then A has inverse $B = A^{-1}$!

(ii) If $\mathrm{rref}([A \mid I_n]) \neq [I_n \mid B]$ then A is not invertible.

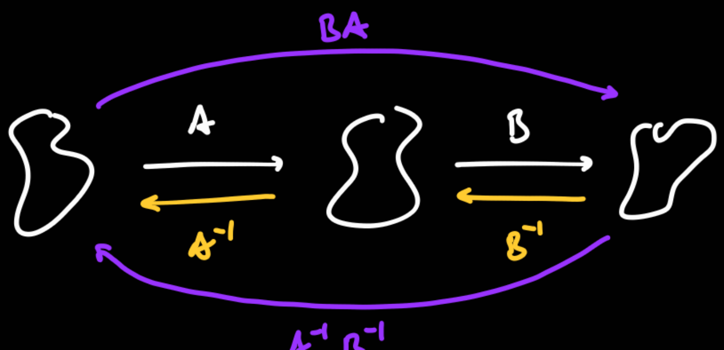
Example: The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$ is invertible:

$$\mathrm{rref}\left(\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

Theorem: If A, B are invertible $n \times n$ matrices, then:

(i) $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

(ii) BA is invertible with inverse $(BA)^{-1} = A^{-1}B^{-1}$.



Example:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta\cdot\cos\theta + (-\sin\theta)\cdot(-\sin\theta) & \cos\theta\cdot\sin\theta + (-\sin\theta)\cdot\cos\theta \\ \sin\theta\cdot\cos\theta + \cos\theta\cdot(-\sin\theta) & \sin\theta\cdot\sin\theta + \cos\theta\cdot\cos\theta \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has <u>determinant</u> $\det(A) = ad - bc$. The matrix A is invertible if and only if $\det(A) \neq 0$. If A is invertible then:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \qquad \left(A^{-1} = \frac{1}{\det(A)}\cdot \operatorname{adj}(A) \right)$$

Example:

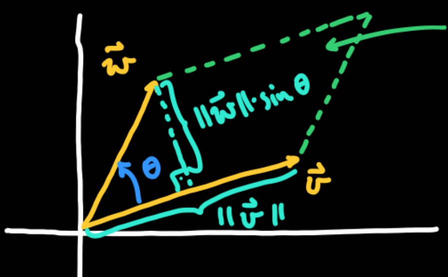
(i) For which values of k is the matrix $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

$\det(A) = (k-5)(k+1)$, and if $\det(A) \neq 0$ then A is invertible.

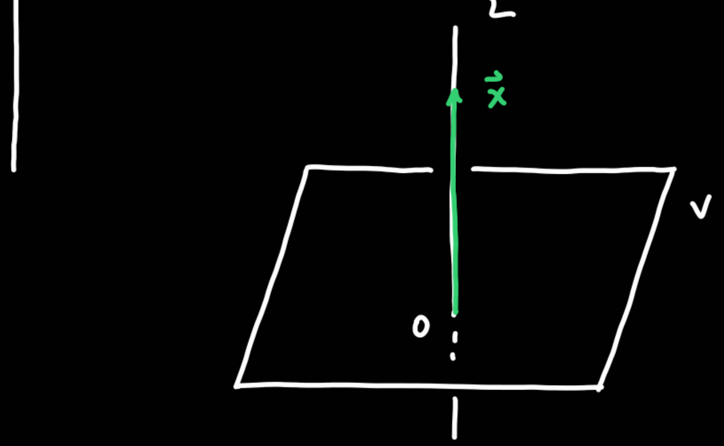
So if $k \neq 5$ or $k \neq -1$, then A is invertible.

Let $\vec{v}, \vec{w}$ in $\mathbb{R}^2$, consider $A = [\vec{v}\ \vec{w}]$, then:

$\det(A) = \|\vec{v}\|\cdot\|\vec{w}\|\cdot\sin(\theta)$, θ the angle between $\vec{v}$ and $\vec{w}$.

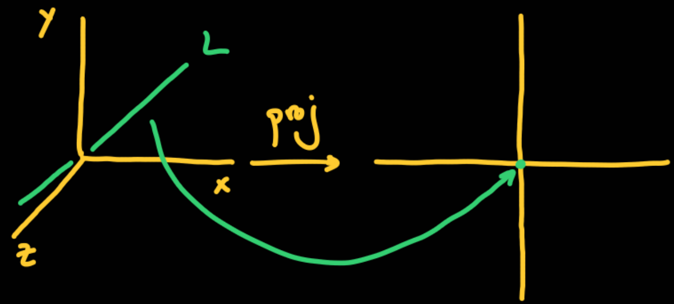


Area = base · height = $\|\vec{v}\|\cdot\|\vec{w}\|\cdot\sin\theta = \det(A)$.



projection onto V:
$\mathbb{R}^3 \longrightarrow V = \mathbb{R}^2$
2×3 matrix

any $\vec{x}$ parallel to L gets projected onto $\vec{0}$ in V.



$\mathbb{R}^2 = V$