2. Subspaces of $\mathbb{R}^n$.

The image of a function T is the set of values it takes.

"belongs to", in

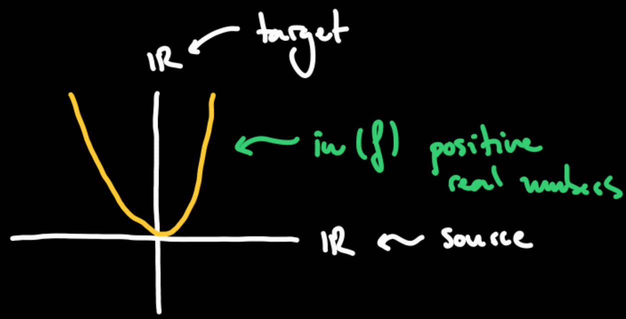
$$\operatorname{im}(T) = \{ T(x) \mid x \in X \} = \{ y \in Y \mid y = T(x) \text{ for some } x \in X \}$$

such that

Examples:

$| \xrightarrow{f} |$

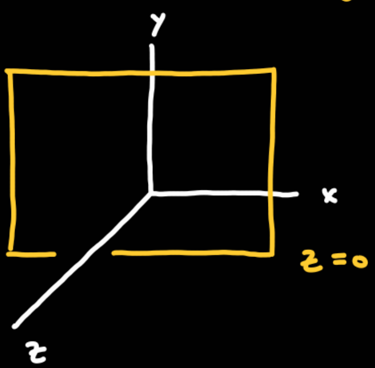
1. Non-linear example: $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$



2. $T : \mathbb{R}^3 \to \underline{\mathbb{R}^3}$ projection onto the x-y plane.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}_{\text{" is living in } \mathbb{R}^2 \text{ "}}$$

$$\operatorname{im}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \;\middle|\; x \in \mathbb{R}, y \in \mathbb{R} \right\} \text{ in } \underline{\mathbb{R}^3}$$

target



source   target

$T : X \longrightarrow Y$

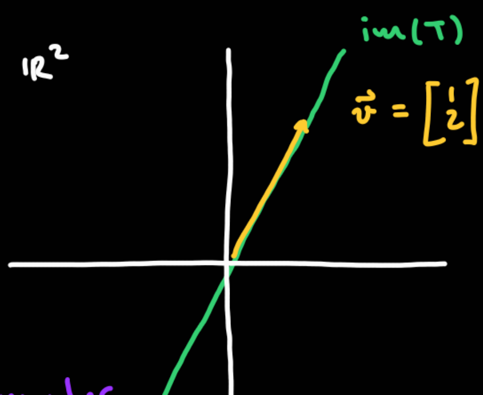
$\begin{bmatrix} 1 \\ 0 \\ z \end{bmatrix}$ is <u>not</u> in $\operatorname{im}(T)$ for $z \neq 0$

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

$$T(\vec{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= \underline{(x_1 + 3x_2)} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So $\operatorname{im}(T) = \left\{ k \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; k \in \mathbb{R} \right\}$.



by changing x_1, x_2, we

can make $x_1 + 3x_2$ into any real number

Conceptual insight: the image of a matrix is a linear combination of its columns.

Let $\vec{v}_1, \dots, \vec{v}_m$ in $\mathbb{R}^n$, the set of all linear combinations of these vectors is called their <u>span</u>:

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \{ c_1\vec{v}_1 + \dots + c_m\vec{v}_m \mid c_1, \dots, c_m \in \mathbb{R} \}.$$

"the image of a linear tansformation is the span of their columns".

<u>Theorem:</u> Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear tansformation. Then:

(i) $\vec{0} \in \text{im}(T)$, $\vec{0} \in \mathbb{R}^n$.

(ii) If $\vec{v}_1, \vec{v}_2 \in \text{im}(T)$ then $\vec{v}_1 + \vec{v}_2 \in \text{im}(T)$.

(iii) If $\vec{v} \in \text{im}(T)$, $k \in \mathbb{R}$, then $k\vec{v} \in \text{im}(T)$.

This will be the definition of "subspace"!

<u>Example:</u> T is given by $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ then $\text{im}(T) = \{ k\begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid k \in \mathbb{R} \}$.

(i) $0 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii) and (iii) also hold.

$T: \mathbb{R}^m \to \mathbb{R}^n$

The <u>kernel</u> of a linear transformation T is the set of values that the function takes to $\vec{0}$.

$$\ker(T) = \{ \vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0} \}$$

these values are the solution to the equation $T(\vec{x}) = \vec{0}$.

<u>Example:</u> Find the kernel of $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

Solving $T(\vec{x}) = \vec{0}$ is solving the system:

$x_1 = x_3$

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + 2x_2 + 3x_3 &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$x_2 = -2x_3$ rref

Set $x_1 = t$ the free variable, then $\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are the solutions.

$$\ker(T) = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; k \in \mathbb{R} \right\}.$$

<u>Remark:</u> The previous theorem is true if we replace $\operatorname{im}(T)$ by $\ker(T)$.

<u>Example:</u> If T is invertible then $\vec{0}$ is the only solution to $T(\vec{x}) = \vec{0}$.

$$\ker(T) = \{\vec{0}\} = 0.$$

<u>Example:</u> If T has $\ker(T) = \{\vec{0}\} = 0$ then $T(\vec{x}) = \vec{0}$ has only one solution.

$A\vec{x} = \vec{0}$

so there are <u>no</u> free variables. So $\operatorname{rank}(A) = m$.

$T : \mathbb{R}^m \to \mathbb{R}^n$, A $n \times m$. rows $\to$ equations

columns $\to$ variables

<u>Theorem:</u> Let A be an $n \times m$ matrix.

(i) $\ker(A) = 0$ if and only if $\operatorname{rank}(A) = m$.

(ii) If $\ker(A) = 0$ then $m \leq n$.

(iii) If $m > n$ then we must have non-zero vectors in $\ker(A)$.

(iv) Let $n = m$, then $\ker(A) = 0$ i.f.o.i.f A invertible.

$A \quad n \times n \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\begin{bmatrix} A \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} (n+k) \times n \longrightarrow \mathbb{R}^n \longrightarrow \mathbb{R}^{n+k}$