A subset W of $\mathbb{R}^n$ is called a <u>linear subspace</u> of $\mathbb{R}^n$ if:

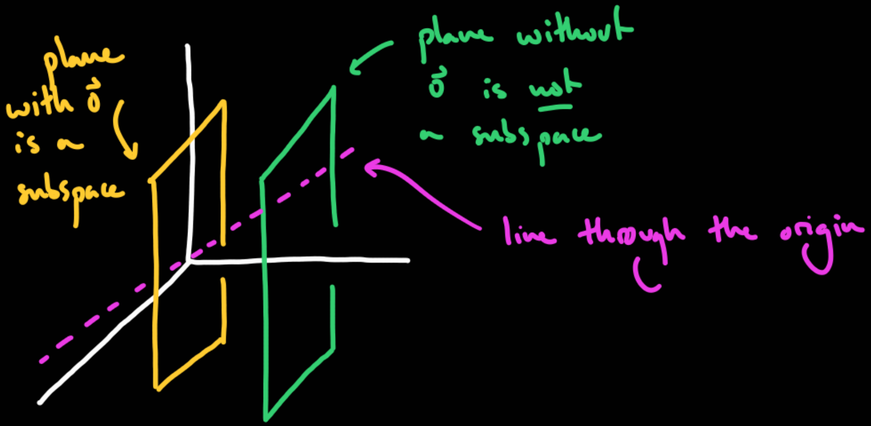
(i) $\vec{0} \in W$

(ii) W is closed under addition $\quad \underset{W}{\vec{v}_1} + \underset{W}{\vec{v}_2} \in W$

(iii) W is closed under scalar multiplication. $\quad k\underset{W}{\vec{v}} \in W$

<u>Examples:</u>

1\.



2\. $T : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}}_{\ker(T)} \qquad \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{\operatorname{im}(T)}$$

3\. V the plane in $\mathbb{R}^3$ given by $x + 2y + 3z = 0$.

(a) Find A with $\ker(A) = V$. $\qquad T : \underset{\text{input}}{\overset{\text{columns}}{\mathbb{R}^m}} \longrightarrow \underset{\text{output}}{\overset{\text{rows}}{\mathbb{R}^n}}$

$A\vec{x} = \vec{0}$ ⟵ we want this, which is $\ker(A)$, to be V, which is $x + 2y + 3z = 0$.

We want A to input a vector in $\mathbb{R}^3$, and to output <u>one</u> equation,

so A has to be a 1×3 matrix. $\qquad A = [a\ b\ c]$

$$0 = A\vec{x} = [a\ b\ c]\begin{bmatrix} x \\ y \\ \vdots \end{bmatrix} = x + 2y + 3z$$

$$0 = A\vec{x} = \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = x + 2y + 3z$$

$ax + by + cz = x + 2y + 3z$ so $a = 1, b = 2, c = 3.$

The matrix $A = [1 \ 2 \ 3]$ has kernel V (given by $x + 2y + 3z = 0$).

(b) B with $\operatorname{im}(B) = V$.

"The image of a matrix is the span of its columns".

B has to have two columns, and they cannot be parallel.

$$B = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \qquad B\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \cdot \vec{v}_1 + \cdots x_m \cdot \vec{v}_m$$

$\vec{v}_i$ has n components

multiplication of matrix by vector

$\operatorname{im}(B) = V$, we want two vectors $\vec{v}_1, \vec{v}_2$ spanning V.

$x + 2y + 3z = 0$

$z = 0$ $(y = 1)$ $\rightsquigarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$y = 0$ $(z = 1)$ $\rightsquigarrow \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

So $B = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$.

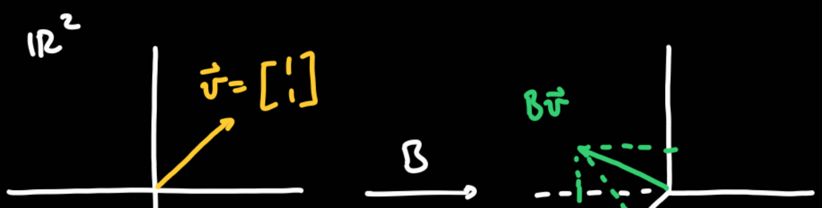
columns

rows

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

$n \times p$ $\quad q \times m$

$p = 2 \quad p = q \quad$ so $q = 2$



$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

<u>Example:</u>

$$\begin{array}{cc} A = & \begin{bmatrix} 1 & 3 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix} \\ & \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \end{array}$$

$\vec{v}_2 = 3 \cdot \vec{v}_1$

$\vec{v}_4 = 2 \cdot \vec{v}_1 + \vec{v}_3$

$\vec{v}_2$ and $\vec{v}_4$ are a linear combination of $\vec{v}_1, \vec{v}_3$.

$$\underset{\substack{\parallel \\ \operatorname{im}(A)}}{\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4)} = \left\{ c_1 \cdot \vec{v}_1 + \underline{c_2} \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 + \underline{c_4} \cdot \vec{v}_4 \right\} =$$

$$= \left\{ c_1 \vec{v}_1 + 3 \cdot c_2 \cdot \vec{v}_1 + c_3 \vec{v}_3 + 2 \cdot c_4 \cdot \vec{v}_1 + c_4 \cdot \vec{v}_3 \right\} =$$

$$= \left\{ (c_1 + 3 \cdot c_2 + 2 \cdot c_4) \cdot \vec{v}_1 + (c_3 + c_4) \cdot \vec{v}_3 \right\} = \operatorname{span}(\vec{v}_1, \vec{v}_3)$$

Let $\vec{v}_1, \dots, \vec{v}_m$. We say that $\vec{v}_i$ is <u>redundant</u> if it is a linear combination of $\vec{v}_1, \dots, \vec{v}_{i-1}$. If none of these vectors are redundant, we call them <u>linearly independent</u>.

Let $\vec{v}_1, \dots, \vec{v}_m$ span V and be linearly independent, we call them a <u>basis</u> of V.

$V = \operatorname{span}(\vec{v}_1, \dots, \vec{v}_m)$ is a linear subspace.

<u>Example:</u> $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. To detect if these vectors are linearly independent, we compute $\ker\left(\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right)$.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longleftarrow \text{computing kernel.}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x_3 = t$$

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{so the kernel is } \operatorname{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right).$$

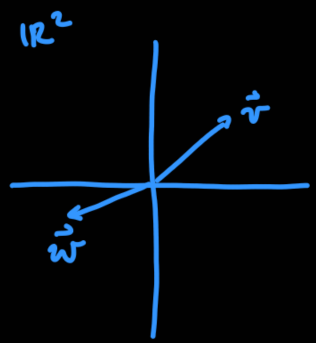
Since we have a kernel:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 2 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = -\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \longleftarrow \vec{v}_3 \text{ is redundant!}$$

A square matrix with kernel zero is invertible.

$$\begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix}$$



$$= x_1 \cdot \vec{v}_1 + \cdots + x_m \cdot \vec{v}_m$$

$$= \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

rows n correspond to equations

columns m correspond to variables