Recall: The vectors $\vec{v}_1, \dots, \vec{v}_m$ form a basis of $V$ a subspace of $\mathbb{R}^n$ when:

(i) $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ and

(ii) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent.

Theorem: Let $\vec{v}_1, \dots, \vec{v}_m$ be vectors in $V$. Then they are a basis if and only if every $\vec{v} \in V$ can be written uniquely as a linear combination:

$$\vec{v} = c_1 \cdot \vec{v}_1 + \dots + c_m \cdot \vec{v}_m$$

coordinates of $\vec{v}$ with respect to $\vec{v}_1, \dots, \vec{v}_m$

Example: $\mathbb{R}^n$ has basis $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ coordinates of $\vec{v}$ in terms of the basis $\vec{e}_1, \dots, \vec{e}_n$.

Note: For $V$ a subspace, a spanning set of $V$ will be larger (or equal) (i.e. will have more elements) than a basis of $V$.

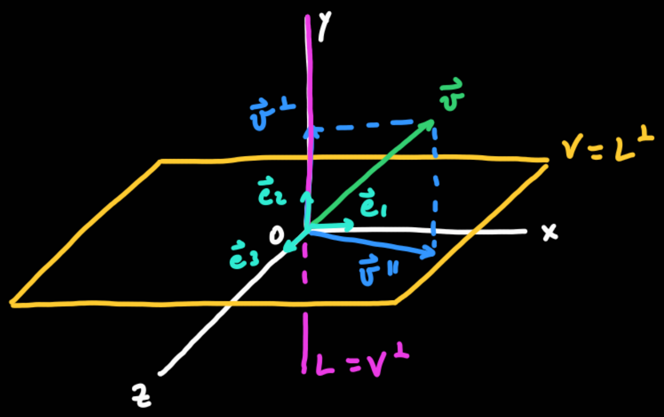
Two different basis of $V$ will have the same number of vectors.

Theorem: Let $V$ be a subspace with a basis $\vec{v}_1, \dots, \vec{v}_m$. We say that $V$ has dimension $m$. Then:

(i) There are at most $m$ linearly independent vectors in $V$.

(ii) We need at least $m$ vectors to span $V$. $\quad V = \text{span}(\vec{w}_1, \dots, \vec{w}_j)$

**Example:** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a projection onto the plane $V$.




$\mathrm{im}(T) = V$ so $V = \mathrm{span}(\vec{w}_1, \vec{w}_2)$

two non-parallel vectors in $V$

so $\dim(V) = 2$.

$\ker(T) = L$ so $L = \mathrm{span}(\vec{n})$

vector perpendicular to $V$

so $\dim(L) = 1$.

Note: $\dim(\mathrm{im}(T)) + \dim(\ker(T)) = 3$

source has dimension 3

$$\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \text{dimension of "source"}$$

**Theorem:** (Rank-Nullity) Let $A$ be an $n \times m$ matrix, then:

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$$

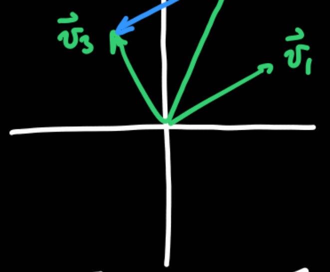
We call $\dim(\ker(A))$ the <u>nullity</u> of $A$:

$$(\text{rank } A) + (\text{nullity of } A) = m$$

**Theorem:** Let $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ be an $n \times m$ matrix.

(i) To construct a basis of $\mathrm{im}(A)$, we pick the columns of $A$ corresponding to the columns of $\mathrm{rref}(A)$ having leading ones.

(ii) The columns of $A$ corresponding to columns of $\mathrm{rref}(A)$ that do <u>not</u> contain leading ones can be used to find a basis of $\ker(A)$.




$\vec{v}_3 = \vec{v}_2 - \vec{v}_1$

$\mathbb{R}^2 = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \text{span}(\vec{v}_1, \vec{v}_2)$

<u>Example:</u> $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$

$$\text{rref}(A) = \begin{array}{c} \begin{array}{ccccc} 1 & & & 4 & \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

columns with leading 1s.

So column 1 and column 4 will be a basis of the image:

$$\text{im}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Moreover:

$$\text{rref}(A) = \begin{array}{c} \begin{array}{ccccc} & 2 & 3 & & 5 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$$

columns without leading 1s

So columns 2, 3, 5 are redundant.

$\vec{v}_2 = 2 \cdot \vec{v}_1 \rightsquigarrow -2\vec{v}_1 + \vec{v}_2 = 0 \rightsquigarrow \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in $\ker(A)$

$\vec{v}_3 = 0 \cdot \vec{v}_1 \rightsquigarrow \vec{v}_3 = 0 \rightsquigarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ in $\ker(A)$

$\vec{v}_5 = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_4 \rightsquigarrow -\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = 0 \rightsquigarrow \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ in $\ker(A)$

Finally:

$$\ker(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$