Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a basis of V: $\quad (V \text{ in } \mathbb{R}^m)$

$\mathbb{R}^n$

$$\vec{x} = c_1 \cdot \vec{v}_1 + \dots + c_m \cdot \vec{v}_m$$

<u>$\mathcal{B}$ - coordinates</u> of $\vec{x}$

$$\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}}$$

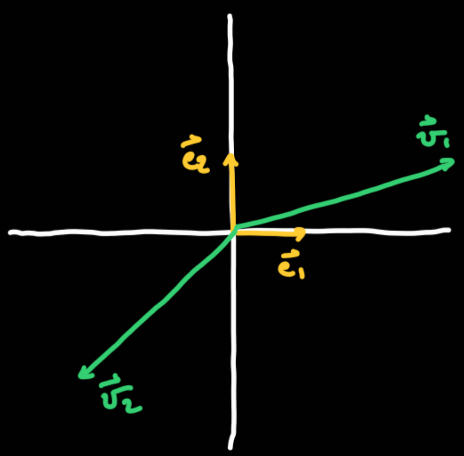
<u>$\mathcal{B}$ - coordinate vector</u> of $\vec{x}$

$$\begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x} = \underbrace{c_1 \cdot \vec{v}_1 + \dots + c_m \vec{v}_m}_{\text{basis } \mathcal{B}} = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{S} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = S\, [\vec{x}]_{\mathcal{B}}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathcal{S} \quad \text{standard basis}: \ \vec{e}_1, \dots, \vec{e}_n}$$

The matrix S is "changing coordinates", it is changing from basis $\mathcal{B}$ to $\mathcal{S}$.

<u>Example:</u> $\mathbb{R}^2$ has basis $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -2 \\ -2 \end{bmatrix}}_{\vec{v}_2} \right\}$.



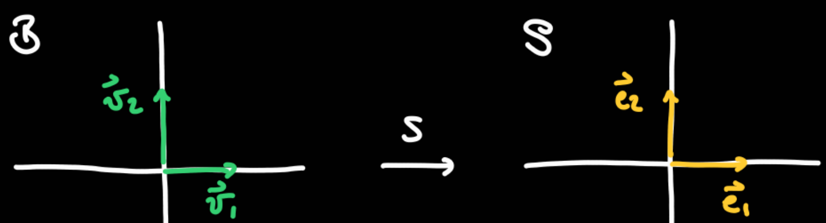
$$S = \begin{bmatrix} 3 & -2 \\ 1 & -2 \end{bmatrix}$$

takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = 1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2$

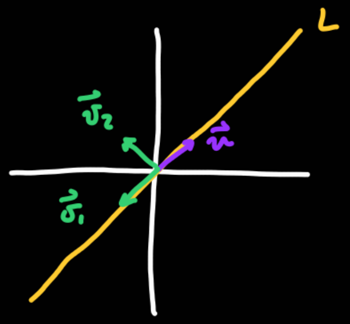
and returns $\begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{S}} = S \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}}$.

$$S : \mathbb{R}^2_{\mathcal{B}} \longrightarrow \mathbb{R}^2_{\mathcal{S}}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}} \xrightarrow{S} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \vec{v}_2 .$$



Example: Projection on a line:



$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}} = c_1 \cdot \vec v_1 + c_2 \cdot \vec v_2$$

Project $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}}$ onto L: $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}_{\mathcal{B}}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}}$$

Theorem: Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation, $\mathcal{B} = \{\vec v_1, \dots, \vec v_n\}$ a basis of $\mathbb{R}^n$, then there exists a matrix B transforming $[\vec x]_{\mathcal{B}}$ into $[T(\vec x)]_{\mathcal{B}}$.

$$B[\vec x]_{\mathcal{B}} = [T(\vec x)]_{\mathcal{B}}.$$

$$B = \begin{bmatrix} | & & | \\ [T(\vec v_1)]_{\mathcal{B}} & \cdots & [T(\vec v_n)]_{\mathcal{B}} \\ | & & | \end{bmatrix} \qquad \begin{bmatrix} | & & | \\ T(\vec e_1) & \cdots & T(\vec e_n) \\ | & & | \end{bmatrix}$$

Example: $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{\mathcal{B}} = [\vec x]_{\mathcal{B}}$, $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_{\mathcal{B}} = [\vec y]_{\mathcal{B}}$ $\qquad \mathcal{B} = \{\vec v_1, \vec v_2\}$

2 equations $\qquad$ 2 equations

$$\underbrace{c_1}_{} \cdot \underset{\uparrow 2}{\vec v_1} + \underbrace{c_2}_{} \cdot \underset{\uparrow 2}{\vec v_2} = \underbrace{\vec x}_{} \qquad \underbrace{d_1}_{} \cdot \vec v_1 + \underbrace{d_2}_{} \cdot \vec v_2 = \underbrace{\vec y}_{}$$

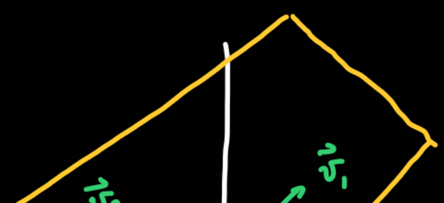
known

How do we find $\vec v_1$ and $\vec v_2$?

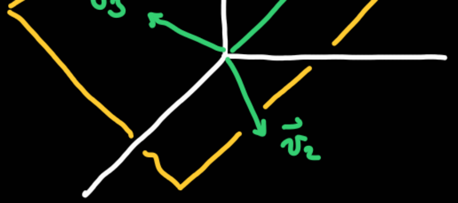
Example: $\mathbb{R}^3$

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec v_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec v_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec v_3} \right\}$$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ orthogonal projection onto $V = \text{span}(\vec v_1, \vec v_3)$



$\vec v = \begin{bmatrix} 1 \\ -1 \\ \cdots \end{bmatrix}$ gives L a perpendicular line



$V \quad x - y + z = 0$

$$\begin{cases} \vec{v} \cdot \vec{v}_1 = 0 \\ \vec{v} \cdot \vec{v}_3 = 0 \end{cases}$$

Working in S then $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$

$$T(\vec{x}) = \text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$$

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix} \quad T(\vec{e}_2) = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \quad T(\vec{e}_3) = \begin{bmatrix} -1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Working in $\mathcal{B}$:

$T(\vec{v}_1) = \vec{v}_1$ since $\vec{v}_1 \in V$

$T(\vec{v}_3) = \vec{v}_3$ since $\vec{v}_3 \in V$

Computing $\text{proj}_V(\vec{v}_2) = T(\vec{v}_2)$, note that $\vec{v}_2 = \underbrace{\frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_3}_{\vec{v}_2^{\parallel}} + \underbrace{\frac{2}{3}\vec{v}}_{\vec{v}_2^{\perp}}$

$$T(\vec{v}_2) = \frac{1}{3}\vec{v}_1 + \frac{1}{3}\vec{v}_3 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u})\vec{u}$$

So the matrix of T in basis $\mathcal{B}$:

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & [T(\vec{v}_2)]_{\mathcal{B}} & [T(\vec{v}_3)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

Note:

$S = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$

$$S = \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array} \qquad \vec{v}_2 \ \vec{v}_3 \ \vec{v}_1$$

$$\begin{tikzcd}
\mathbb{R}^3, S \arrow[r, "A"] \arrow[d, "S^{-1}"'] & \mathbb{R}^3, S \\
\mathbb{R}^3, \mathcal{B} \arrow[r, "B"] & \mathbb{R}^3, \mathcal{B} \arrow[u, "S"']
\end{tikzcd}$$

<u>Check:</u> $A = S B S^{-1}$