Recall: A vector $\vec{v}$ has length $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

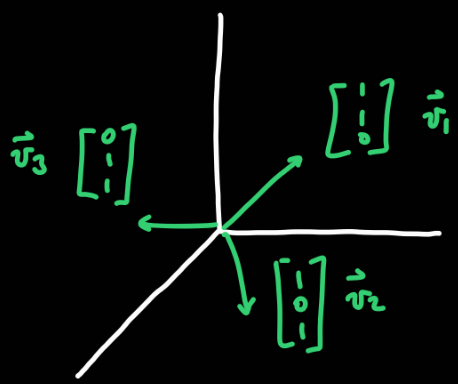
$\|\vec{u}\| = 1 \leftarrow$ unit vector

Perpendicular: $\vec{v}\cdot\vec{w} = 0$

A vector $\vec{v}$ is orthogonal to a subspace W if $\vec{v}\cdot\vec{w} = 0$ for all $\vec{w}$ in W.

Remark: $\vec{v}$ is orthogonal to W if $\vec{v}$ is orthogonal to a basis of W.

Example:



These vectors are not orthogonal.

$\|\vec{v}_i\| = \sqrt{1+1} = \sqrt{2}$

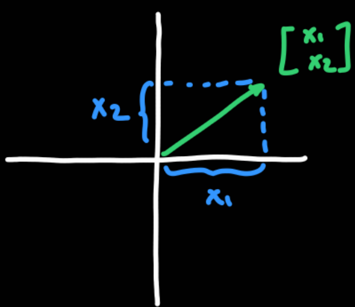
Example: Given a plane $ax + by + cz = 0$ (d), why is $\vec{x} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ orthogonal to the plane? Because if $\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ is in the plane, then:

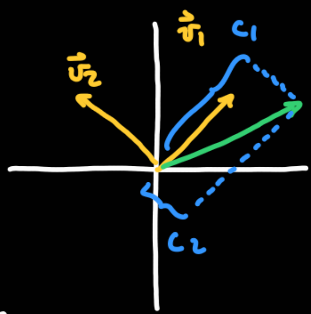
$a\cdot v_1 + b\cdot v_2 + c\cdot v_3 = 0$

$$\vec{x}\cdot\vec{v} = \begin{bmatrix}a & b & c\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = av_1 + bv_2 + cv_3 = 0$$

Motivation:

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\cdot\vec{e}_1 + x_2\cdot\vec{e}_2$$





$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\cdot\vec{v}_1 + c_2\cdot\vec{v}_2$$

The vectors $\vec{v}_1, \dots, \vec{v}_m$ are said to be orthonormal if they all have unit size, and

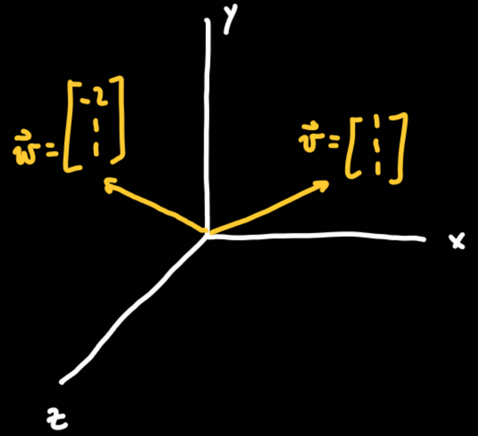
they are all perpendicular to each other.

Theorem:

(i) Orthonormal vectors are linearly independent.

(ii) If $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^n$ are orthonormal then $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$.

Example: We can make any two orthogonal vectors into an orthonormal basis:



$\vec{w} \cdot \vec{v} = -2 + 1 + 1 = 0.$

$\vec{u} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$ is a vector perpendicular to both.

So $\vec{w}$ and $\vec{v}$ are in the plane $3y - 3z = 0.$

$y - z = 0$

$0 \cdot (-2) + 3 \cdot 1 - 3 \cdot 1 = 0$

$0 \cdot 1 + 3 \cdot 1 - 3 \cdot 1 = 0$

$$\vec{v}\,' = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{w}\,' = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}\,' = \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

This $\mathcal{B} = \{\vec{u}\,', \vec{v}\,', \vec{w}\,'\}$ is an orthonormal basis of $\mathbb{R}^3$.

$\mathcal{R} = \{\vec{v}\,', \vec{w}\,'\}$ is an orthonormal basis of $3y - 3z = 0.$

Key idea: projecting can be used to find / compute perpendicular vectors

V a subspace : $\vec{x} = \underbrace{\vec{x}^{\,\parallel}}_{\text{in } V} + \underbrace{\vec{x}^{\,\perp}}_{\text{orthogonal to } V}$

If $\vec{u}_1, \dots, \vec{u}_m$ are an orthonormal basis of V, then:

$$\vec{x}^{\parallel} = \mathrm{proj}_V(\vec{x}) = \underbrace{(\vec{x}\cdot\vec{u}_1)}_{c_1}\vec{u}_1 + \cdots + \underbrace{(\vec{x}\cdot\vec{u}_m)}_{c_m}\vec{u}_m$$

<u>Example:</u> Consider $\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}$ an orthonormal basis of $\underbrace{y-z=0}_{V}$.

$\vec{x} = \begin{bmatrix}4\\5\\6\end{bmatrix}$:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$\vec{x}^{\parallel} = \mathrm{proj}_V(\vec{x}) = \Bigg(\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}}_{\vec{u}_1\cdot x}\Bigg)\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{u}_1} + \left(\frac{1}{\sqrt{6}}\begin{bmatrix}-2 & 1 & 1\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}5\\5\\5\end{bmatrix} + \begin{bmatrix}-1\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}4\\11/2\\11/2\end{bmatrix} \qquad 0\cdot 4 + \frac{11}{2} - \frac{11}{2} = 0$$

$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix}4\\5\\6\end{bmatrix} - \begin{bmatrix}4\\11/2\\11/2\end{bmatrix} = \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix}$ which lies in the same line as $\begin{bmatrix}0\\3\\-3\end{bmatrix}$

$$\vec{x} = \begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4\\11/2\\11/2\end{bmatrix} + \begin{bmatrix}0\\-1/2\\1/2\end{bmatrix} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$