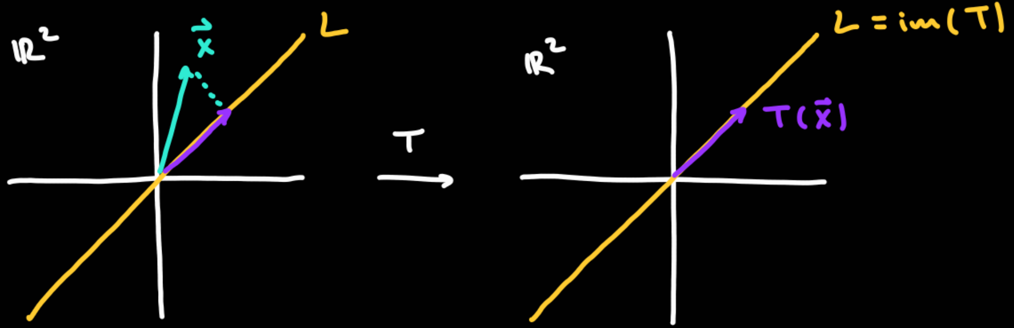
<u>Recall:</u> If $\vec{u}_1, \dots, \vec{u}_n$ is an orthonormal basis of $\mathbb{R}^n$ then:

$$\vec{x} = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{x}\cdot\vec{u}_n)\vec{u}_n$$

Let V be a subspace of $\mathbb{R}^n$, the <u>orthogonal complement</u> $V^\perp$ of V is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to all vectors in V.

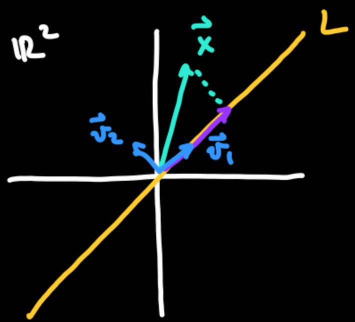
$$V^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x}\cdot\vec{v} = 0 \text{ for all } \vec{v} \in V\}.$$

If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the orthogonal projection onto V, then any vector $\vec{x} \in V^\perp$ will be sent to zero: $T(\vec{x}) = \vec{0}$. The converse is also true. Thus $V^\perp = \ker(T)$.



T projection onto V $\qquad V^\perp = \ker(\mathrm{proj}_V)$.

<u>Short digression into reflections:</u>



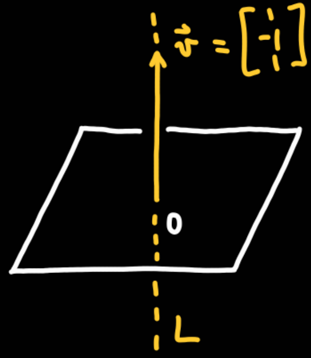
$$\mathrm{ref}_L(\vec{x}) = \vec{x} - 2\vec{x}^\perp$$

$T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = -\vec{v}_2$ $\qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

<u>Example:</u> Consider $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the orthogonal projection onto the plane V spanned

by $\begin{bmatrix} \\ \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \\ \\ 1 \end{bmatrix}$. The associated matrix to this transformation is:

$$\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} = A.$$



$$A\vec{x} = \vec{0} \sim \left[\begin{array}{ccc|c} 2/3 & 1/3 & -1/3 & 0 \\ 1/3 & 2/3 & 1/3 & 0 \\ -1/3 & 1/3 & 2/3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{l} x - z = 0 \\ y + z = 0 \\ z = t \end{array} \qquad \vec{x} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right) = L.$$

<u>Theorem:</u> Let V be a subspace of $\mathbb{R}^n$. Then:

(i) The orthogonal complement $V^\perp$ is also a subspace of $\mathbb{R}^n$.

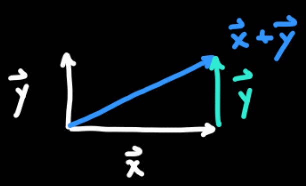
(ii) The intersection of V and $V^\perp$ is $\vec{0}$. $\qquad \underbrace{\vec{v}}_{V} \cdot \underbrace{\vec{v}}_{V^\perp} = 0$

(iii) $\dim(V) + \dim(V^\perp) = n$

(iv) $(V^\perp)^\perp = V$

<u>Theorem:</u> Let $\vec{x}, \vec{y}$ be vectors in $\mathbb{R}^n$, let θ be the angle between them, let V be any subspace of $\mathbb{R}^n$. Then:

(i) $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x}$ and $\vec{y}$ are orthogonal.



(ii) $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$ with equality if and only if $\vec{x}$ is in V.

(iii) (Cauchy - Schwartz) $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\cdot\|\vec{y}\|$

with equality if and only if $\vec{x}$ and $\vec{y}$ are parallel.

(iv) $$\cos(\theta) = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\cdot\|\vec{y}\|}$$

$|\cos(\theta)| \leq 1$

<u>Example:</u> $\mathfrak{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ basis of $\mathbb{R}^3$. This is <u>not</u> an orthonormal basis:

(i) These vectors have length $\sqrt{2}$.

(ii) The dot product of any two of the tree vectors is 1, so:

$$\cos(\theta) = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \quad \text{so} \quad \theta = \frac{\pi}{3}, \text{ which is } \underline{\text{not}} \ \frac{\pi}{2}.$$

$$\left\| \begin{bmatrix}1\\1\\0\end{bmatrix} \right\| = \sqrt{\begin{bmatrix}1\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}} = \sqrt{1+1} = \sqrt{2}$$

<u>Idea:</u> normalize and take projections.