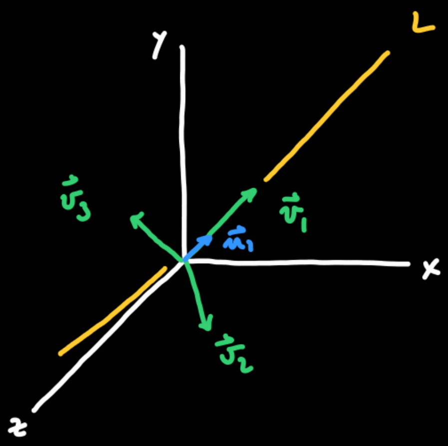
Example: $\mathfrak{B} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$.

$\qquad\qquad\quad \vec{v}_1 \qquad \vec{v}_2 \qquad \vec{v}_3$

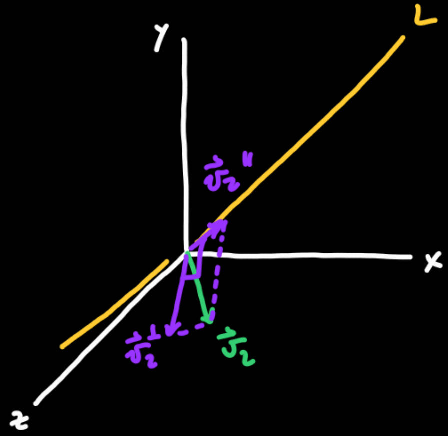
$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

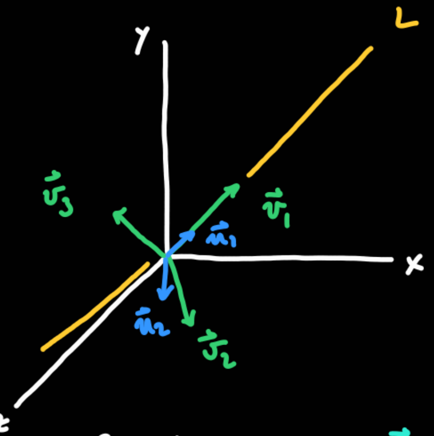


$$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$$

$$\vec{v}_2^{\parallel} = \operatorname{proj}_L(\vec{v}_2) = (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \left(\begin{bmatrix}1 & 0 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \right) \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$





$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{2}{\sqrt{6}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

Remark: $\quad \vec{v} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$

$$\operatorname{span}(\vec{u}_1, \vec{u}_2) = \operatorname{span}(\vec{v}_1, \vec{v}_2) = V$$

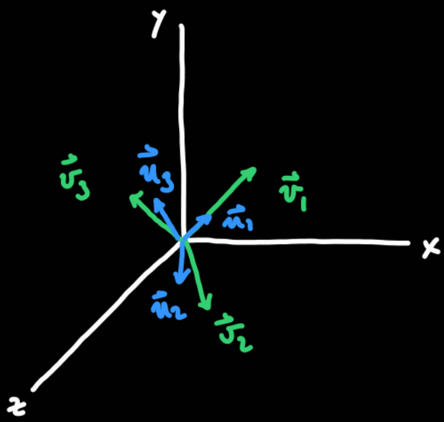
$$\vec{v}_3 = \vec{v}_3^{\parallel} + \vec{v}_3^{\perp}$$

One way of doing this: project onto $\vec{u}_1$, project onto $\vec{u}_2$, and this sum is the component $\vec{v}_3{}^{\parallel}$.

$$\vec{v}_3{}^{\perp} = \mathrm{proj}_{V^{\perp}}(\vec{v}_3) = \left(\vec{v}_3 \cdot \frac{\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|} = \left(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \frac{-2}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} =$$

$$= \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3{}^{\perp}}{\|\vec{v}_3{}^{\perp}\|} = \frac{1}{\sqrt{3}} \cdot \frac{3}{2} \cdot \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \frac{\vec{v}}{\|\vec{v}^{\perp}\|}$$



Gram-Schmidt process:

$\vec{v}_1, \dots, \vec{v}_m$ basis of V

Decompose $\vec{v}_j$ into the parallel and perpendicular components with respect to span$(\vec{v}_1, \dots, \vec{v}_{j-1})$. $\quad \vec{v}_j = \vec{v}_j{}^{\parallel} + \vec{v}_j{}^{\perp}$

Then:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{u}_2 = \frac{\vec{v}_2{}^{\perp}}{\|\vec{v}_2{}^{\perp}\|}, \quad \dots, \quad \vec{u}_m = \frac{\vec{v}_m{}^{\perp}}{\|\vec{v}_m{}^{\perp}\|}$$

$$\{\vec{v}_1, \dots, \vec{v}_m\} \xrightarrow[\text{Gram-Schmidt process}]{} \{\vec{u}_1, \dots, \vec{u}_m\}$$

$$\mathcal{B} \longrightarrow \mathcal{S} = \{\vec{e}_1, \dots, \vec{e}_n\}$$

$$\mathcal{B} = \{\vec{w}_1, \dots, \vec{w}_n\} \qquad \begin{bmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_m \\ | & & | \end{bmatrix} = S$$

$\mathbb{R}^n$ $\qquad\qquad\qquad\qquad$ $\mathbb{R}^n$

The matrix switching from basis $\{\vec{v}_1, \dots, \vec{v}_m\}$ to basis $\{\vec{u}_1, \dots, \vec{u}_m\}$ is given via the QR factorization.

M $n \times m$ $\qquad\qquad M = QR$

Q: orthonormal columns

R: upper triangular with (strictly) positive diagonal entries

$$\{\vec{u}_1, \dots, \vec{u}_m\} \qquad Q = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}$$

If R is square, is R invertible? Yes.

Example: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\qquad M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

$M = QR$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \qquad QR = \left[Q \begin{bmatrix} r_{11} \\ 0 \\ 0 \end{bmatrix} \quad Q \begin{bmatrix} r_{12} \\ r_{22} \\ 0 \end{bmatrix} \quad Q \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \right]$$

$$r_{11} = \vec{u}_1 \cdot \vec{v}_1 = \sqrt{2} \qquad r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}$$

$$r_{13} = \vec{u}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \qquad r_{22} = \vec{u}_2 \cdot \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}} \qquad r_{33} = \vec{u}_3 \cdot \vec{v}_3 = \frac{2}{\sqrt{3}}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$QR = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M$$

$$\underset{n\times n}{A} = S^{-1} \underset{n\times n}{B}\, S$$

$$B = S^{-1} A S$$

$$\underset{n\times m}{M} = \underset{n\times m}{Q}\,\underset{m\times m}{R}$$