A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is <u>orthogonal</u> if it preserves lengths.

$$\|T(\vec{x})\| = \|\vec{x}\| \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n$$

<u>Theorem:</u> If $T: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear transformation and $\vec{x}, \vec{y}$ are orthogonal vectors, then $T(\vec{x})$ and $T(\vec{y})$ are orthogonal.

<u>Theorem:</u>

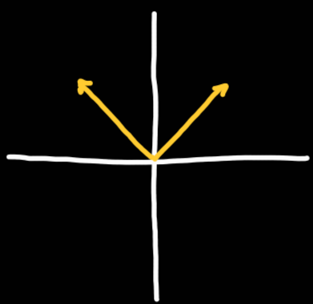
(i) A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if $\{T(\vec{e}_1), \dots, T(\vec{e}_n)\}$ is an orthonormal basis.

(ii) A matrix A is orthogonal if and only if its columns are an orthonormal basis.

(iii) If A and B are orthogonal then AB is also orthogonal.

(iv) If A is orthogonal then A^{-1} is orthogonal.

<u>Examples:</u>



$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A^{-1} = A = A^T$$

$$B = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} \qquad B^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} = B^T$$

$$M = \begin{bmatrix} 1 & 1 & 0 \\ \dots \end{bmatrix} \qquad Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ \dots \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \dots \end{bmatrix} = Q^T$$

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} = Q^T$$

Let A be an $m \times n$, the <u>transpose</u> of A, denoted by A^T, is the $n \times m$ matrix whose ij-th entry is the ji-th entry of A.

<u>Example:</u> $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ 3×2 $\qquad$ 2×3 $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

A square matrix A is said to be <u>symmetric</u> if $A = A^T$, and it is said to be <u>skew symmetric</u> if $A = -A^T$.

<u>Example:</u> $A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} = A$

$B = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$ $\qquad B^T = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix} = -B$

<u>Question:</u> Can a skew symmetric matrix have non-zero entries in the diagonal?

<u>Theorem:</u> A matrix $\underset{n \times n}{A}$ is orthogonal if and only if $A^T A = I_n$. (i.e. $A^{-1} = A^T$)

<u>Orthogonal matrix algebra:</u> / <u>Transpose matrix algebra:</u>

A, B matrices of the appropriate sizes, invertible when necessary. Then:

(i) $(A + B)^T = A^T + B^T$

(ii) $(kA)^T = k A^T$

(iii) $(AB)^T = B^T A^T$

(iv) $\text{rank}(A^T) = \text{rank}(A)$

(v) $(A^T)^{-1} = (A^{-1})^T$

<u>The matrix of an orthogonal projection:</u>

V subspace of $\mathbb{R}^n$

$\vec{u}_1, \dots, \vec{u}_m$ orthonormal basis of V

The matrix of the orthogonal projection onto V is:

$$P = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_m & - \end{bmatrix}}_{Q^T}$$

<u>Question:</u> Explain why P is symmetric.

<u>Example:</u> $T: \mathbb{R}^3 \to \mathbb{R}^3$ the orthogonal projection onto $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{w}} \right\}$$

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}^{\perp} = \vec{w} - (\vec{w} \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\vec{\ } \quad \vec{w}^{\perp} \quad [-1$

$$u_2 = \frac{\quad}{\|\vec{w}^{\perp}\|} = \frac{\quad}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$V = \operatorname{span}\left(\vec{v}, \vec{w}\right) = \operatorname{span}(\vec{u}_1, \vec{u}_2)$$

$$P = \begin{bmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{bmatrix} \begin{bmatrix} - & \vec{u}_1 & - \\ \\ - & \vec{u}_2 & - \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \operatorname{proj}_V(\vec{e}_1) & \operatorname{proj}_V(\vec{e}_2) & \operatorname{proj}_V(\vec{e}_3) \end{array} \end{array}$$