$A \quad n \times m \qquad A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

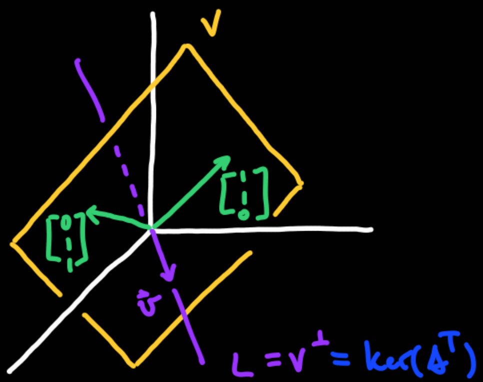
$A^T \quad m \times n \qquad A^T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$A^T A : \mathbb{R}^m \longrightarrow \mathbb{R}^m \qquad (MN)^T = N^T M^T$

$(\mathrm{im}(A))^\perp = \ker(A^T)$

<u>Example:</u> $T : \mathbb{R}^3 \to \mathbb{R}^3$ projection onto $V = \mathrm{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$.

$$A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \qquad A^T = A$$



$\mathrm{im}(A)$ is the plane $x - y + z = 0$

$\vec{v} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\mathrm{im}(A) = V$

$V^\perp = L$

So to have $\ker(A^T) = (\mathrm{im}(A))^\perp = V^\perp = L$.

To compute $\ker(A^T)$ we solve $A^T \vec{x} = \vec{0}$:

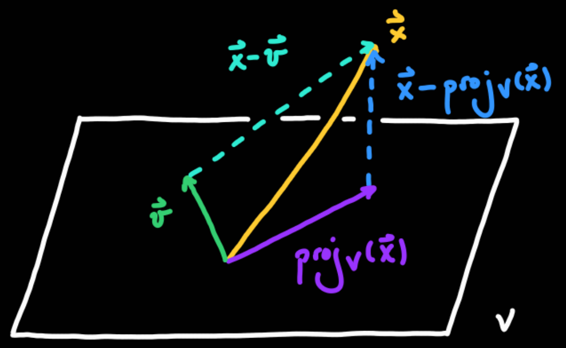
$$\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = t \cdot \vec{v}$$

So $\ker(A^T) = \mathrm{Span}(\vec{v}) = L = (\mathrm{im}(A))^\perp$.

$\mathbb{R}^3 \qquad \dim(\mathrm{im}(A)) = 2$

$\mathbb{R}^n$ $\quad \dim(\mathrm{im}(A)) = n-1$

<u>Review:</u>



$\|\vec{x}-\mathrm{proj}_V(\vec{x})\|$

$\|\vec{x}-\vec{v}\|$

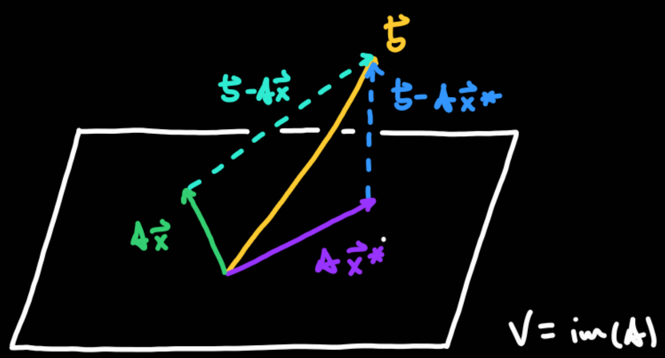
The orthogonal projection onto a subspace V is solving a minimization problem:

it looks at all the distances $\|\vec{x}-\vec{v}\|$ for $\vec{v}$ in V, and chooses the smallest.

$$\|\vec{x}-\mathrm{proj}_V(\vec{x})\| \leq \|\vec{x}-\vec{v}\| \text{ for } \vec{v} \text{ in } V$$

Suppose we are given $A\vec{x}=\vec{b}$, A is an $n\times m$ matrix, $\vec{b}$ vector in $\mathbb{R}^n$.

$$\vec{b}-A\vec{x}=\vec{0}$$



We call $\vec{x}^*$ a <u>least-squares solution</u> of the system if $\|\vec{b}-A\vec{x}^*\| \leq \|\vec{b}-A\vec{x}\|$ for any other $\vec{x}$ in $\mathbb{R}^m$.

Note that from $A\vec{x}=\vec{b}$ we can get $A^TA\vec{x}=A^T\vec{b}$, which is always a consistent system. This equation is called the <u>normal equation</u> of the system.

Theorem: If A $(n\times m)$ has kernel $\{\vec{0}\}$ then the system $A\vec{x}=\vec{b}$ has exactly

one least-squares solution: $\vec{x}^* = \underbrace{(A^TA)^{-1}}_{A^{-1}(A^T)^{-1}}A^T\vec{b}$.

(If A orthogonal then $A^TA = I_n$.) as a matrix

Example: $T:\mathbb{R}^3\to\mathbb{R}^3$ projection onto $V=\operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

$$P=\begin{bmatrix}2/3 & 1/3 & -1/3\\ 1/3 & 2/3 & 1/3\\ -1/3 & 1/3 & 2/3\end{bmatrix}$$

A has two linearly independent columns, so $\ker(A)=\{\vec{0}\}$.

Find the least squares solution of $A\vec{x}=\vec{b}$ for $A=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\0\\1\end{bmatrix}$:

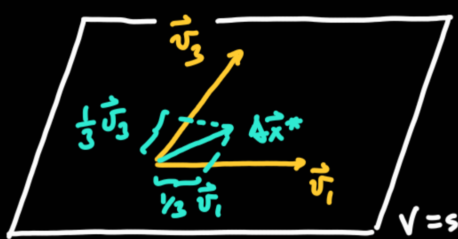
(we want $V=\operatorname{im}(A)$, and $\operatorname{im}(A)$ is the span of its columns)

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \left(\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=$$

$$=\begin{bmatrix}2&1\\1&2\end{bmatrix}^{-1}\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}$$

"decomposing $\vec{x}^*$ into $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$"

$$\operatorname{im}(A)=\operatorname{span}\left(\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1},\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\vec{v}_3}\right)$$



$V=\operatorname{span}(\vec{v}_1,\vec{v}_3)$

$$A\vec{x}^* = \begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{3}\vec{v}_1+\frac{1}{3}\vec{v}_3$$

$\operatorname{proj}_V(\vec{v}_2)=\frac{1}{3}\vec{v}_1+\frac{1}{3}\vec{v}_3 \qquad \vec{v}_2=\vec{b}=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

Theorem: V be a subspace of $\mathbb{R}^n$ with basis $\vec{v}_1,\dots,\vec{v}_m$, the matrix associated to the orthogonal projection onto V is:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \left(\begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_m & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} \right)^{-1} \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_m & - \end{bmatrix}.$$

Do this with $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$.

$$P = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

true by the Theorem

$$P = A(A^TA)^{-1}A^T$$