Denote $M_n(\mathbb{R})$ the set of all $n\times n$ matrices, a <u>determinant</u> is a function:

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R} \quad \text{satisfying:}$$

(i) It is linear with respect to columns:

$$\det\begin{bmatrix} | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i + \vec{c}_i{}' & \cdots & \vec{c}_n \\ | & & | & & | \end{bmatrix} = \det\begin{bmatrix} | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i & \cdots & \vec{c}_n \\ | & & | & & | \end{bmatrix} + \det\begin{bmatrix} | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i{}' & \cdots & \vec{c}_n \\ | & & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} | & & | & & | \\ \vec{c}_1 & \cdots & k\vec{c}_i & \cdots & \vec{c}_n \\ | & & | & & | \end{bmatrix} = k\cdot\det\begin{bmatrix} | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i & \cdots & \vec{c}_n \\ | & & | & & | \end{bmatrix}$$

$$\det\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = = \det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \det\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(ii) It is alternating in the columns:

$$\det\begin{bmatrix} | & & | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i & \cdots & \vec{c}_i & \cdots & \vec{c}_n \\ | & & | & & | & & | \end{bmatrix} = 0$$

$$\begin{bmatrix} | & & | & & | & & | \\ \vec{c}_1 & \cdots & \vec{c}_i & \cdots & \vec{c}_j & \cdots & \vec{c}_n \\ | & & | & & | & & | \end{bmatrix} \quad \vec{c}_i = \vec{c}_j$$

(iii) The determinant of the identity matrix is 1:

$$\det\begin{bmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_n \\ | & & | \end{bmatrix} = 1$$

$$\det\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$2\cdot\det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det\cdots \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$2\cdot\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\right)$$

$$\det\cdots \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

<u>Example:</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det(A) = ad - bc \quad$ A invertible if and only if $\det(A) \neq 0$.

$$\det\begin{bmatrix} a+a' & b \\ c+c' & d \end{bmatrix} = (a+a')d - b(c+c') = ad - bc + a'd - bc' = \det(A) + \det\begin{bmatrix} a' & b \\ c' & d \end{bmatrix}$$

$$\det\begin{bmatrix} a' & b \\ c' & d \end{bmatrix} = a'd - bc'$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$

Example: $A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Example: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$.

$$\det(A) = a_{11}\cdot a_{22}\cdots a_{n-1\,n-1}\cdot a_{nn}$$

This same formula holds for lower triangular matrices.

Theorem: A $n\times n$ invertible matrix, if when computing ref(A) we swap rows s times and we divide rows by the scalars $k_1,\dots,k_r$ then:

$$\det(A) = (-1)^s\cdot k_1\cdots k_r$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{\frac{1}{-4}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$$

$$\xrightarrow[R_3+3R_2]{R_1-2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2-2R_1]{R_1+R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\det(A) = -4\cdot 2 = -8$$

$$\det(A) = 1\cdot2\cdot2 + 2\cdot1\cdot2 + 3\cdot3\cdot1 - 3\cdot2\cdot2 - 1\cdot1\cdot1 - 2\cdot3\cdot2 = -8$$

Let A be an $n\times n$ matrix, the $(n-1)\times(n-1)$ matrix A_{ij} obtained from A by removing the i-th row and j-th column is called a <u>submatrix</u> of A. The determinant $\det(A_{ij})$ is called a <u>minor</u> of A.

<u>Theorem:</u>

Expand by j-th column: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$

Expand by i-th row: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$

<u>Example:</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Expand down column 2:

$$\det(A) = (-1)^{1+2} \cdot 2 \cdot \det\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + (-1)^{2+2} \cdot 2 \cdot \det\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} + (-1)^{3+2} \cdot 1 \cdot \det\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} =$$

$$= -2 \cdot 4 + 2 \cdot (-4) - 1 \cdot (-8) = -8$$