Recall:

$$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$A \longmapsto \det(A)$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad j\text{-th column}$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad i\text{-th row}$$

Remark: The determinant is symmetric with respect rows and columns.

$$A \qquad A^T = A \qquad \det(A) = \det(A^T)$$

(i) The determinant is linear with respect to each row:

$$\det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i + \vec{r}_i{}' & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = \det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} + \det \begin{array}{c} \text{column} \\ \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i{}' & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} \end{array}$$

$$\det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & k\vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = k \det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix}$$

(ii) The determinant is alternating with respect to rows:

$$\det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_i & - \\ & \vdots & \\ - & \vec{r}_i & - \\ - & \vec{r}_n & - \end{bmatrix} = 0$$

(iii) The determinant of the identity is 1:

$$\det \begin{bmatrix} - & \vec{e}_1 & - \\ & \vdots & \\ - & \vec{e}_n & - \end{bmatrix} = 1$$

In particular $\det(A) = \det(A^T)$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ \dots \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 & 2 \\ \dots \end{bmatrix}$

Example: Let $A = \begin{bmatrix} \cdots & \cdots & \cdots \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ so $A^T = \begin{bmatrix} \cdots & \cdots & \cdots \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, now:

$$\det(A^T) = (-1)^{1+1} \cdot 1 \cdot \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + (-1)^{2+1} \cdot 2 \cdot \det \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} + (-1)^{3+1} \cdot 3 \cdot \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} =$$

$$= 1 \cdot (+3) - 2 \cdot 4 + 3 \cdot (-1) = -8 = \det(A).$$

Recall: We can compute determinants by reducing the matrix to rref. We pick up a sign every time that we permute two rows, and we pick up the scalars that we multiply / divide each row to obtain leading 1's.

It is useful to remember how elementary operations affect the determinant:

(i) If B is obtained from A by dividing a row of A by k then:

$$\det(B) = \frac{1}{k} \cdot \det(A).$$

(ii) If B is obtained from A by swapping two rows, then:

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by adding a multiple of a row to a different row, then:

$$\det(B) = \det(A).$$

Example: $A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$ $\quad \det(A) = -7$

(i) $B = \begin{bmatrix} 1/8 & 5/8 \\ 2 & 3 \end{bmatrix}$ $\quad \det(B) = \frac{1}{8} \cdot 3 - 2 \cdot \frac{5}{8} = \frac{3-10}{8} = \frac{-7}{8} = \frac{1}{8} \cdot \det(A)$

B is obtained by dividing the first row of A by 8.

(ii) $B = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$ $\det(B) = 2\cdot5 - 1\cdot3 = 7 = -\det(A)$

(iii) $B = \begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix}$ $\det(B) = 3\cdot3 - 2\cdot8 = 9-16 = -7 = \det(A)$

$B = \begin{bmatrix} 1+2 & 5+3 \\ 2 & 3 \end{bmatrix}$

$$\det(B) = \det\begin{bmatrix} 1+2 & 5+3 \\ 2 & 3 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} + \det\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = \det(A)$$

Theorem: A square matrix is invertible if and only if its determinant is not zero.

Theorem: A invertible then $\det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}$!

$$A \xrightarrow[\text{swaps}]{\text{multiplying rows}} \mathrm{rref}(A) \qquad I_n \longrightarrow A^{-1} \qquad A^{-1} \xrightarrow{\text{swap}} \mathrm{rref}(A^{-1}) \qquad [A \mid I_n] \xrightarrow{\mathrm{rref}} [I_n \mid A^{-1}]$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

$R_2 - 3R_1$, $R_3 - 2R_1$ $\quad \frac{1}{-4}R_2$ $\quad R_1 - 2R_2$, $R_3 + 3R_2$ $\quad \frac{1}{2}R_3$ $\quad R_1 + R_3$, $R_2 - 3R_3$

$$A \longrightarrow I_n$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} -\frac{3}{8} & \frac{1}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{8} & -\frac{3}{8} & \frac{1}{2} \end{bmatrix}}_{A^{-1}}$$

$$\det(A^{-1}) = \frac{1}{-8} = \frac{1}{\det(A)}.$$

WARNING: We would like that $\det(A+B) = \det(A) + \det(B)$. Not true.

$\det(kA) = k \cdot \det(A)$. <u>Not</u> true.

$$\det\left(I_n - I_n\right) \stackrel{?}{=} \det(I_n) + \det(I_n)$$

<u>Theorem:</u>

(i) $\det(kA) = k^n \cdot \det(A)$

(ii) $\det(AB) = \det(A)\det(B)$

(iii) $\det(A^m) = \det(A)^m$

(iv) If A and B are similar then $\det(A) = \det(B)$.

$AS = SB$

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{A} \text{ is } \underline{\text{not}} \text{ similar to } \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{B}$$

but both have determinant 1.