Question: What is the determinant of an orthogonal matrix?

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has determinant -1. **Answer:** ± 1

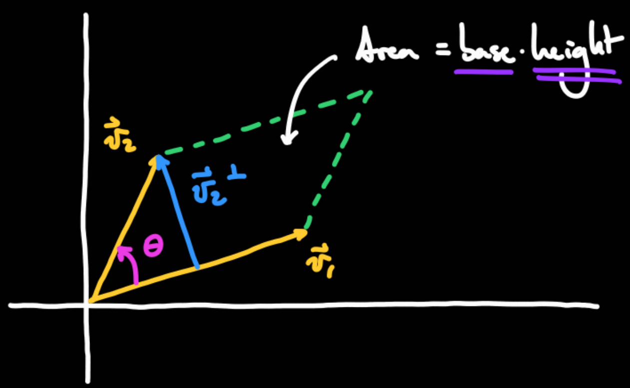
$$A^T A = I_n \longrightarrow \det(A^T A) = \det(I_n) \longrightarrow \overbrace{\det(A^T)}^{\det(A) \text{ by symmetry of det}} \det(A) = 1$$

$$\longrightarrow \det(A)\det(A) = 1 \longrightarrow (\det(A))^2 = 1.$$

so $\det(A) = \pm 1$.

Geometric interpretation of the determinant:

$A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$ is a 2×2 matrix, then $\det(A) = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin(\theta)$ where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$.



$\|\vec{v}_2^{\perp}\| = \|\vec{v}_2\| \cdot \sin(\theta)$

$|\det(A)| = \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| = \text{Area}$

$|\det(A)|$ is the area of the parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2$.

$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ is an $n\times n$ matrix, invertible, so by Gram-Schmidt A has a QR decomposition: $A = \underbrace{Q}_{\text{orthogonal}}\overbrace{R}^{\text{upper triangular with positive entries in the diagonal}}$

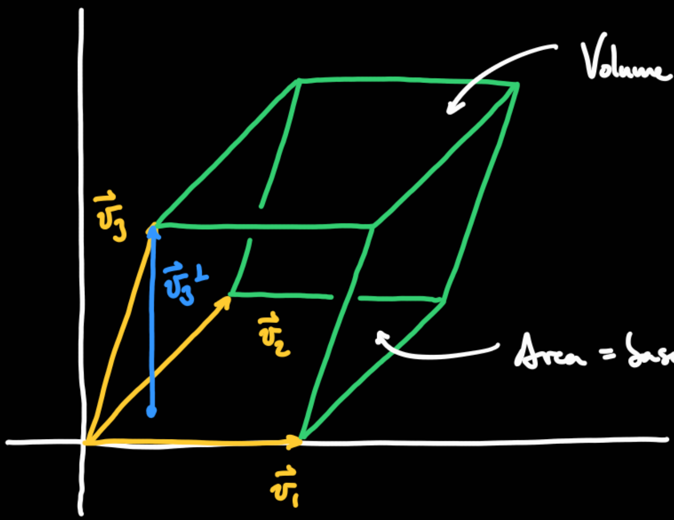
$$|\det(A)| = |\det(QR)| = |\det(Q)| \cdot |\det(R)| = r_{11} \cdot r_{22} \cdots r_{nn} = \underbrace{\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_n^{\perp}\|}.$$

$r_{11} = \|\vec{v}_1\|, \quad r_{22} = \|\vec{v}_2^{\perp}\|, \ldots, \quad r_{nn} = \|\vec{v}_n^{\perp}\|$

Generalizes the formula:
Area = base · height.

$\vec{v}_i^{\perp}$ is the component of $\vec{v}_i$ perpendicular to $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_{i-1})$.

Case $n=3$:



Volume = base · height = $\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdot \|\vec{v}_3^{\perp}\|$

Area = base · height = $\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\|$

The volume in $\mathbb{R}^n$ of the linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ is $\det\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$.

We can also use determinants to find solutions of linear systems:

__Theorem:__ (Cramer's rule) Let $A\vec{x} = \vec{b}$, if A is an invertible $n \times n$ matrix then

the solution has i-th component:

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}$$

where $A_{\vec{b},i}$ is the matrix obtained when we replace the i-th column of A by $\vec{b}$.

The __classical adjoint__ of an invertible matrix A, denoted $\mathrm{adj}(A)$, is the matrix:

$$(\mathrm{adj}(A))_{ij} = (-1)^{i+j} \cdot \det(A_{ji}).$$

__Theorem:__ $A^{-1} = \dfrac{\mathrm{adj}(A)}{\det(A)}$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$, to compute A^{-1} we first compute all minors $\det(A_{ij})$

and put them in their respective positions in a 3x3 matrix:

$$\begin{bmatrix} \overset{+}{(-1)^{1+1}} \det(A_{11}) & \overset{-}{(-1)^{1+2}} \det(A_{12}) & \overset{+}{(-1)^{1+3}} \det(A_{13}) \\ \overset{-}{(-1)^{1+2}} \det(A_{21}) & \overset{+}{(-1)^{2+2}} \det(A_{22}) & \overset{-}{(-1)^{2+3}} \det(A_{23}) \\ \overset{+}{(-1)^{1+3}} \det(A_{31}) & \overset{-}{(-1)^{3+2}} \det(A_{32}) & \overset{+}{(-1)^{3+3}} \det(A_{33}) \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & -4 & 3 \\ -4 & 8 & -4 \end{bmatrix}$$

now we take the transpose:

$$\begin{bmatrix} 3 & -1 & -4 \\ -4 & -4 & 8 \\ -1 & 3 & -4 \end{bmatrix} = \operatorname{adj}(A)$$

finally, we divide by $\det(A) = -8$:

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} -3/8 & 1/8 & 4/8 \\ 4/8 & 4/8 & -1 \\ 1/8 & -3/8 & 4/8 \end{bmatrix}.$$

$$\det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = \vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)$$

$$\|\vec{v}_2 \times \vec{v}_3\| = \|\underset{\substack{\uparrow \\ \operatorname{span}(\vec{v}_1).}}{\vec{v}_2^{\perp}}\| \cdot \|\underset{\substack{\uparrow \\ \operatorname{span}(\vec{v}_1, \vec{v}_2).}}{\vec{v}_3^{\perp}}\|$$