non-zero   can be zero

<u>Recall:</u> $\vec{v}$ is called an <u>eigenvector</u> of a matrix A if $A\vec{v} = \underbrace{\lambda}_{\text{real scalar}} \vec{v}$.

The scalar λ is called an <u>eigenvalue</u> of A.

$$A\vec{v} - \lambda\vec{v} = \vec{0} \qquad \underbrace{(A - \lambda I_n)}_{\text{det may be } 0}\vec{v} = \vec{0} \qquad (A - \lambda I_n)\vec{x} = \vec{0}$$

Let A be an $n \times n$ matrix, a real number λ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$. This equation is called the <u>characteristic equation</u> of A. Seeing λ as a variable, $\det(A - \lambda I_n)$ is a polynomial in degree n, called the <u>characteristic polynomial</u> of A, denoted $f_A(\lambda)$.

<u>Examples:</u> $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ being sent to $\underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{\text{scaling of } 2}, \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{scaling of } 0}$.

$$f_A(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 1 = \lambda^2 + 1 - 2\lambda - 1 =$$

$$= \lambda^2 - 2\lambda = \lambda(\lambda - 2).$$

The solutions of the characteristic equation (which are the same as the roots of the characteristic polynomial) are $\lambda = 0$ and $\lambda = 2$.

Thus the eigenvalues of A are 2 and 0.

Let A be an $n \times n$ matrix. The <u>trace</u> of A is the sum of its diagonal entries, denoted $tr(A)$.

entries, denoted $tr(A)$.

<u>Example:</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has characteristic polynomial:

$$f_A(\lambda) = \det(A - \lambda I_2) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc =$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc) = \lambda^2 - tr(A)\lambda + \det(A)$$

Let A be an $n \times n$ matrix, let λ_0 be an eigenvalue of A. The <u>algebraic multiplicity</u> of λ_0 is the number of times it appears on the characteristic polynomial. This is the largest k such that:

$$f_A(\lambda) = (\lambda - \lambda_0)^k \cdot g(\lambda) \quad \text{with} \quad g(\lambda_0) \neq 0.$$

<u>Example:</u> $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, find all eigenvalues with algebraic multiplicities.

$$f_A(\lambda) = \det\begin{bmatrix} 2/3-\lambda & 1/3 & -1/3 \\ 1/3 & 2/3-\lambda & 1/3 \\ -1/3 & 1/3 & \frac{2}{3}-\lambda \end{bmatrix} \overset{(*)}{=} -\lambda^3 + 2\lambda^2 - \lambda = \underbrace{-\lambda}_{0} \cdot \underbrace{(\lambda-1)^2}_{1}$$

The eigenvalues of A are $\lambda = 0$, $\lambda = 1$ with multiplicities 1 and 2 respectively.

$$(*) = (2/3-\lambda)(2/3-\lambda)(\tfrac{2}{3}-\lambda) + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{-1}{3}\cdot(2/3-\lambda)\cdot(\tfrac{2}{3}-\lambda)$$

$$- \frac{-1}{3}\cdot(\frac{2}{3}-\lambda)\cdot\frac{-1}{3} - \frac{1}{3}\cdot\frac{1}{3}\cdot(\frac{2}{3}-\lambda) - (\frac{2}{3}-\lambda)\cdot\frac{1}{3}\cdot\frac{1}{3}$$

<u>Theorem:</u> An $n \times n$ matrix will have at most n eigenvalues, counted with multiplicity.

If n is odd we have at least one real eigenvalue. If n is even we may not have real eigenvalues.

Example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The characteristic polynomial is:

$$f_A(\lambda) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1,$$ which does not have any real roots.

Thus A has no real eigenvalues, and so no eigenvectors.

Theorem: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ listed with their algebraic multiplicity then:

$$\det(A) = \lambda_1 \cdots \lambda_n \quad \text{and} \quad \operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n.$$