Recall: $A\vec{v} = \lambda\vec{v} \rightsquigarrow A\vec{v} - \lambda\vec{v} = \vec{0} \rightsquigarrow \underbrace{(A - \lambda I_n)}\vec{v} = \vec{0}$

eigenvectors are in the kernel of $A - \lambda I_n$.

The *eigenspace* of λ is the kernel of $A - \lambda I_n$, denoted E_λ.

$$E_\lambda = \ker(A - \lambda I_n) = \{ \vec{v} \in \mathbb{R}^n \mid A\vec{v} = \lambda\vec{v} \}.$$

Example: Find the eigenspaces of $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

$\lambda = 0 \quad \lambda = 1$ $\qquad f_A(\lambda) = -\lambda \cdot (\lambda - 1)^2$

To find E_0 we compute $\ker(A)$, so we solve $A\vec{x} = \vec{0}$.

$$\vec{x} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad E_0 = \ker(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

To find E_1 we compute $\ker(A - I_3)$, so we solve $(A - I_3)\vec{x} = \vec{0}$.

$$\vec{x} = \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad E_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

The *geometric multiplicity* of λ is the dimension of E_λ, $\operatorname{gemu}(\lambda)$.

$$\operatorname{gemu}(\lambda) = \dim\left(\ker(A - \lambda I_n)\right) = \operatorname{nullity}(A - \lambda I_n) = n - \operatorname{rank}(A - \lambda I_n).$$

Let A be an $n \times n$ matrix. An *eigenbasis* of A is a basis of $\mathbb{R}^n$ such that every element of the basis is an eigenvector of A.

Example:

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenbasis $\mathfrak{S} = \{\vec{e}_1, \vec{e}_2\}$

2. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenbasis $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ has eigenbasis $\mathfrak{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{v}_2}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does not have an eigenbasis.

$A\vec{v}_1 = \vec{v}_1$

$A\vec{v}_2 = \vec{v}_2$

$A(c_1\vec{v}_1 + c_2\vec{v}_2) =$

$A c_1\vec{v}_1 + A c_2\vec{v}_2 =$

$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 =$

$(c_1\vec{v}_1 + c_2\vec{v}_2)$

If $\mathfrak{B}$ is an eigenbasis for A then the linear transformation associated to A in the basis $\mathfrak{B}$ is diagonal with diagonal entries the eigenvalues of the elements in $\mathfrak{B}$.

Remark:

(a) Find a basis for each E_λ. Put all of those vectors next to each other:

$\vec{v}_1, \dots, \vec{v}_s$, then s is the sum of the geometric multiplicities.

(b) The vectors $\vec{v}_1, \dots, \vec{v}_s$ are linearly independent.

(c) The vectors $\vec{v}_1, \dots, \vec{v}_s$ are an eigenbasis of $\mathbb{R}^n$ if and only if $s = n$.

<u>Theorem:</u> Let A be an $n\times n$ matrix with n distinct eigenvalues. Then there is an eigenbasis of A, and to construct it we find an eigenvector for each eigenvalue.

<u>Theorem:</u> Let A be similar to B, then:

(a) $f_A(\lambda) = f_B(\lambda)$.

(b) $\operatorname{rank}(A) = \operatorname{rank}(B)$ and $\operatorname{null}(A) = \operatorname{null}(B)$.

(c) The eigenvalues, their algebraic and geometric multiplicities, of A and B coincide.

(d) $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$.

<u>Example:</u> $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$ has eigenvalues with different algebraic and geometric multiplicities.

<u>Theorem:</u> $\operatorname{gemu}(\lambda) \leq \operatorname{almu}(\lambda)$.