<u>Example:</u> Find eigenbasis and diagonal matrices similar to:

1. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

<u>Comment:</u> The determinant of a triangular matrix is the product of its diagonal entries.

$\det(A) = 1 \cdot 2 = 2$

$\det(A) = \lambda_1 \cdot \lambda_2$

$2 = (-1) \cdot (-2)$

1. $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

$f_A(\lambda) = (1-\lambda)(2-\lambda) \rightsquigarrow \lambda = 1 \;,\; \lambda = 2$

For the eigenbasis, we need their respective eigenvectors.

" If an $n \times n$ matrix has $n$ distinct eigenvalues then it has an eigenbasis"!

Because of this, $A$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$E_1$ : $\ker(A - I_2)$ $\quad \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \vec{x} = \vec{0}$ $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a solution.

$E_2$ : $\ker(A - 2I_2)$ $\quad \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\quad \vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a solution.

$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$

2. $A = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{2\times 2 \text{ matrix}}$ $\quad f_A(\lambda) = (1-\lambda)^2 \quad \lambda = 1$

$E_1$: $\ker(A - I_2)$ $\quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x} = \vec{0} \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ so $\dim(E_1) = 1$.

Since $\underbrace{\text{the sum of the geometric multiplicities}}_{\text{only } 1}$ does <u>not</u> add up to 2,

$A$ does not have an eigenbasis.

3. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

<u>Example:</u> Find values $a, b, c$ for which $A = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

$f_A(\lambda) = \underbrace{(1-\lambda)}_{1}\underbrace{(-\lambda)}_{0}\underbrace{(1-\lambda)}_{1} \quad \lambda = 1 \quad \lambda = 0$

$\dim(E_0) = 1$

<u>Note:</u> $\underbrace{\text{gemu}(\lambda)}_{1} \leq \underbrace{\text{almu}(\lambda)}_{1}$

$\lambda = 0$

$E_0$ always has one non-zero vector,

so $E_0$ has the span of that vector.

So $E_0$ has at least dim. 1. But $E_0$ has at most dim. 1.

$\dim(E_1) = \text{gemu}(1) = 3 - \text{rank}\begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$

$\text{gemu}(\lambda) = n - \text{rank}(A - \lambda I_n)$

$$A - I_3 = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rank ?}}$$

Answer(s): $2 \leftarrow a = 0 = c, \; b = 1$

$1 \leftarrow a = b = c = 0$

Depends on $a$ and $c$ ⊛ and $b$.

$$\text{rank} \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} a & b \\ -1 & c \end{bmatrix}$$

(a) If $ac + b = 0$ then rank is 1.

(b) If $ac + b \neq 0$ then rank is 2.

(a) $\dim(E_1) = \text{gemu}(1) = 3 - 1 = 2$ $\quad \text{gemu}(0) + \text{gemu}(1) = 3$

(b) $\dim(E_1) = \text{gemu}(1) = 3 - 2 = 1$ $\quad \text{gemu}(0) + \text{gemu}(1) = 2$

Now $A$ has an eigenbasis if and only if $ac + b = 0$.

Remark:

1. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is similar to $\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$.
2. Any matrix with eigenvalues $a \pm ib$ is similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.