6. Symmetric matrices and quadratic forms

Finding eigenbasis.

We say that a matrix is <u>diagonalizable</u> if it has an eigenbasis.

We are now interested in finding eigenbasis that are also orthonormal.

We say that a matrix is <u>orthogonally diagonalizable</u> if it has an orthonormal eigenbasis.

$A \qquad \mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \qquad S \qquad\qquad A = SBS^{-1}$

if $\mathfrak{B}$ is orthonormal then S is orthogonal so $A = SBS^T$

<u>Theorem:</u> (Spectral Theorem) A matrix is orthogonally diagonalizable if and only if it is symmetric.

<u>Remark:</u> $\Rightarrow$) If A is orthogonally diagonalizable then A is symmetric.

$A = SBS^T$, B diagonal $\qquad\qquad A = A^T$

$$A^T = (A)^T = (SBS^T)^T = (S^T)^T B^T S^T = SBS^T = A.$$

$$(MN)^T = N^T M^T \qquad (MN)^{-1} = N^{-1} M^{-1}$$

$$M: \mathbb{R}^n \to \mathbb{R}^m$$

$$M^T: \mathbb{R}^m \to \mathbb{R}^n$$

$$M^{-1}: \mathbb{R}^m \to \mathbb{R}^n$$

Recall: Two eigenvectors with different eigenvalues are linearly independent.

Theorem: Let A be a symmetric matrix, $\vec{v}_1$ and $\vec{v}_2$ have different eigenvalues λ_1 and λ_2, then they are orthogonal. In particular, they are linearly independent.

Proof: We want $\vec{v}_1 \cdot \vec{v}_2 = 0$.

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos(\theta)$$

$$\vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + \cdots + a_n b_n$$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \vec{v}_1^T \vec{v}_2$$

$$v_1^T A \vec{v}_2$$

(a) $\vec{v}_1^T (A \vec{v}_2) = \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2 \vec{v}_1^T \vec{v}_2$

(b) $(\vec{v}_1^T A) \vec{v}_2 = (\vec{v}_1^T A^T) \vec{v}_2 =$

$$= (A \vec{v}_1)^T \vec{v}_2 = (\lambda_1 \vec{v}_1)^T \vec{v}_2 = \lambda_1 \vec{v}_1^T \vec{v}_2$$

So :

$$\lambda_2 \vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T A \vec{v}_2 = \lambda_1 \vec{v}_1 \cdot \vec{v}_2$$

Rmk: linearly dependent means $\vec{v}_1 = k \cdot \vec{v}_2$ so $\vec{v}_1$ and $\vec{v}_2$ have the same eigenvalue:

$$A \vec{v}_2 = \frac{1}{k} \cdot k A \vec{v}_2 =$$

$$= \frac{1}{k} A k \vec{v}_2 =$$

$$= \frac{1}{k} A v_1 = \frac{1}{k} \lambda \vec{v}_1 =$$

$$= \frac{1}{k} \lambda k \vec{v}_2 = \lambda \vec{v}_2.$$

$$\underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} \vec{v}_1 \cdot \vec{v}_2 = 0 \quad \text{so} \quad \vec{v}_1 \cdot \vec{v}_2 = 0. \qquad \square.$$

Method to find orthonormal eigenbasis of symmetric matrices.

1. Find eigenvalues and eigenspaces.

2. Find a basis of each eigenspace. Then use Gram-Schmidt to find an orthonormal eigenbase for that eigenspace.

3. Concatenate all of them.

Example: Find an orthonormal eigenbasis for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 2 \qquad \lambda_2 = 0$$

1. It is a basis. (linear independence).

2. The vectors have length 1. ($\|\vec{u}_1\| = 1 = \|\vec{u}_2\|$).

3. The vectors are orthogonal ($\vec{u}_1 \cdot \vec{u}_2 = 0$).

Example: Find an orthonormal eigenbasis for $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

$$\lambda = 1 \qquad \lambda = 0$$

$$\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad \vec{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\vec{v}_2^{\perp}$

$$\vec{u}_2 = \frac{}{\|\vec{v}_2^{\perp}\|}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \text{proj}_{\text{span}(\vec{u}_1)}(\vec{v}_2) = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \cdots = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$