A function $q(x_1, \dots, x_n)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is called a quadratic form if it is a linear combination of products $x_i x_j$ for $i, j \in \{1, \dots, n\}$.

A quadratic form can be written as:

$$q(\vec{x}) = \vec{x}^T A \vec{x} \quad \text{for a unique symmetric matrix } A.$$

matrix associated to q.

Example: Consider the function:

$$q(x_1, x_2, x_3) = \frac{2}{3}(x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 - x_1 x_3)$$

is this a quadratic form? Yes, because it is a linear combination of products of $x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3$. What is the associated matrix A?

$$\underbrace{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}}_{\vec{x}^T} \underbrace{A}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\frac{2}{3}(x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 - x_1 x_3)}_{q(\vec{x})}$$

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \qquad A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$A = (a_{ij})_{i,j}$

a_{ii} the coefficient of x_i^2

$a_{ij} = a_{ji}$ half the coefficient of $x_i x_j$, $i \neq j$.

Let A be a symmetric matrix, this determines $q(\vec{x}) = \vec{x}^T A \vec{x}$ a quadratic form.

When we apply the Spectral Theorem to A, we obtain:

$\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ an orthonormal eigenbasis with eigenvalues $\lambda_1, \dots, \lambda_n$.

Writing $\vec{x}$ in terms of $\mathfrak{B}$, we find: $\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, so:

$$q(\vec{x}) = \vec{x}^T A \vec{x} = \dots = (c_1\vec{v}_1 + \dots + c_n\vec{v}_n)\cdot(c_1\lambda_1\vec{v}_1 + \dots + c_n\lambda_n\vec{v}_n) =$$

$$= \lambda_1 c_1^2 + \dots + \lambda_n c_n^2.$$

<u>Example:</u> Consider the quadratic form given by $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

Is $x_1 = x_2 = x_3 = 0$ a local/global minimum/maximum or neither?

Eigenvalues: $1, 1, 0$. $\quad$ A is <u>not</u> positive definite.

$q(x_1, x_2, x_3) = c_1^2 + c_2^2 \geq 0$ $\quad$ A is positive semidefinite.

$x_1 = x_2 = x_3 = 0$ we have $q(0,0,0) = 0$.

So $x_1 = x_2 = x_3 = 0$ is a global minimum.

Working over $\mathfrak{B}$, we can have c_3 to be any real number, and as long as

$c_1 = c_2 = 0$, then $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ will be sent to zero.

Let $q(\vec{x}) = \vec{x}^T A \vec{x}$ be a quadratic form, with A a symmetric $n \times n$ matrix.

We say that A is <u>positive definite</u> if $q(\vec{x}) > 0$ for $\vec{x} \neq 0$.

We say that A is <u>positive semidefinite</u> if $q(\vec{x}) \geq 0$ for all $\vec{x}$.

We say that A is <u>indefinite</u> if q takes positive and negative values.

<u>Theorem:</u> A symmetric matrix is positive definite if and only if all its eigenvalues are positive.

Let A be a symmetric $n\times n$ matrix. Set $A^{(i)}$ the $i\times i$ matrix obtained from A by deleting all the rows and columns after the i-th one. These are called <u>principal submatrices</u> of A.

$i\in\{1,\dots,n\}$

<u>Theorem:</u> A symmetric matrix A is positive definite if and only if the determinants of all its principal submatrices are positive.

<u>Example:</u> Consider the quadratic form given by $A=\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

Is A positive definite?

$$\det(A^{(1)}) = \det\left[\tfrac{2}{3}\right] = \frac{2}{3} > 0$$

$$\det(A^{(2)}) = \det\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3} > 0$$

$$\det(A^{(3)}) = \det\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} = 0$$ is <u>not</u> positive.

Thus A is <u>not</u> positive definite.

<u>Rmk:</u> If A is <u>not</u> invertible then $\ker(A)\neq\{\vec{0}\}$, so there is some

non-zero $\vec{v} \in \ker(A)$, so:

$$q(\vec{v}) = \vec{v}^T A \vec{v} = \vec{v}^T \vec{0} = \vec{0}.$$

So A is <u>not</u> positive definite.