<u>Claim:</u> For every linear transformation we have a basis such that:

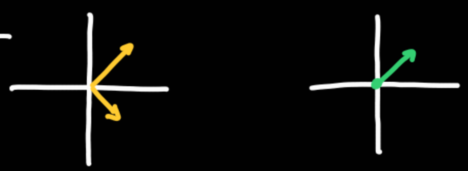
$A \quad n \times m$

$\mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad\qquad \mathbb{R}^m$

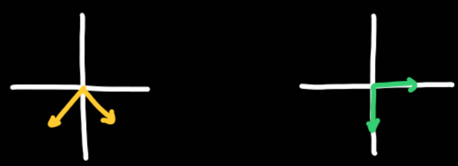
1. The basis is orthogonal.

2. The image of the basis is orthogonal.

<u>proj:</u>



<u>rotations:</u>



<u>Example:</u> Consider the linear transformation given by $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find an ~~orthonormal~~ basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ such that $L(\vec{v}_1)$ and $L(\vec{v}_2)$ are orthogonal.

<u>Hint:</u> This is an application of the Spectral Theorem.

We need a symmetric matrix. $A^T A$

$$A^T A = S D S^T \qquad\qquad A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\ker(A^T A - \lambda I_2)$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}, \quad \lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

$$A^T A \vec{x} = \lambda \cdot \vec{x}$$

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{5}}{2} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \\ 1 \end{bmatrix}$$

orthogonal

$A^T A$ 2×2 $\mathbb{R}^2 \to \mathbb{R}^2$

orthogonal

Apply $L(\vec{v}_1)$ and $L(\vec{v}_2)$. Are they orthogonal? $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$L(\vec{x}) = A\vec{x}$.

$$\underbrace{L(\vec{v}_1) = \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{bmatrix} \qquad L(\vec{v}_2) = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{bmatrix}}_{\text{orthogonal.}}$$

Make $\vec{v}_1, \vec{v}_2$ unitary, check that $L(\vec{u}_1)$ is perpendicular to $L(\vec{u}_2)$.

$\vec{u}_1, \vec{u}_2$

$$\vec{u}_1 = \frac{\sqrt{2}}{\sqrt{3-\sqrt{5}}} \begin{bmatrix} \frac{-1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{1+\frac{2}{\sqrt{5}}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} \qquad \vec{u}_2 = \frac{\sqrt{2}}{\sqrt{3+\sqrt{5}}} \begin{bmatrix} \frac{-1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{1-\frac{2}{\sqrt{5}}} \\ \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix}$$

$$L(\vec{u}_1)\cdot L(\vec{u}_2) = (A\vec{u}_1)\cdot(A\vec{u}_2) = (A\vec{u}_1)^T(A\vec{u}_2) = \vec{u}_1^T A^T A \vec{u}_2 =$$

$$= \vec{u}_1^T(\lambda_2 \vec{u}_2) = \lambda_2 \vec{u}_1^T \vec{u}_2 = \lambda_2 \cdot (\vec{u}_1 \cdot \vec{u}_2) = \lambda_2 \cdot 0 = 0.$$

$$\|L(\vec{u}_1)\|^2 = \frac{3}{2} + \frac{\sqrt{5}}{2} = \lambda_1 \quad \longrightarrow \quad \|L(\vec{u}_1)\| = \sqrt{\lambda_1}.$$

$$\|L(\vec{u}_2)\|^2 = \frac{3}{2} - \frac{\sqrt{5}}{2} = \lambda_2 \qquad \|L(\vec{u}_2)\| = \sqrt{\lambda_2}.$$

The singular values of A are the square roots of the eigenvalues of $A^T A$.

Theorem: Singular value decomposition: Any matrix $\underset{n\times m}{A}$ can be decomposed as:

$$A = U \Sigma V^T$$

orthogonal matrix

orthogonal matrix

$V^{-1} = V^T \quad m \times m$

$U^{-1} = U^T \quad n \times n$

Σ is "diagonal", with entries the singular values of A.

$n \times m$

<u>Method:</u> A is $n \times m$, $\text{rank}(A) = r$

Find $\vec{v}_1, \dots, \vec{v}_m$ orthonormal eigenbasis of $A^T A$.

$\lambda_1 \quad \lambda_m$

$\underbrace{\sigma_1}_{r} \quad \underbrace{\sigma_m}_{m-r}$

non 0   0

$\sigma_i = \sqrt{\lambda_i}$

uppercase Σ sigma

lowercase σ

$\vec{u}_1, \dots, \vec{u}_r$

$\vec{u}_i = \frac{1}{\sigma_i} A \vec{v}_i$

$\vec{u}_{r+1}, \dots, \vec{u}_n$

$$A = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_m & - \end{bmatrix}$$