To do:

1. Bases, notation, how to find them, linear transformation.

2. Drawing sketches for least-squares solutions.

3. Kernel and image of a matrix.

0. Finding zeroes of a quadratic form.

$q(x_1, \dots, x_n)$, we want to find all $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ such that $q(\vec{x}) = 0$.

Let A be the matrix associated to q.

$$q(\vec{x}) = \vec{x}^T A \vec{x}$$

$0 = q(\vec{x}) = \vec{x}^T A \vec{x}$, so if $A\vec{x} = \vec{0}$ then $q(\vec{x}) = 0$.

$\vec{x} \cdot A\vec{x} = 0$ so we want the vectors $A\vec{x}$ such that $\vec{x}$ is perpendicular to $\vec{x}$. So if $\vec{y}$ is in $\text{im}(A)$ and $\vec{y}$ is perpendicular to $\vec{x}$, then there is a chance that $q(\vec{x}) = 0$.

Method:

1. Find all vectors in $\text{im}(A)$.

2. Find all vectors in $\text{im}(A)$ and perpendicular to $\vec{x}$.

3. Find which of these could be written as $A\vec{x}$

3. Find which of those are obtained as $A\vec{x}$.

8.2.22: On the surface: $-x_1^2 + x_2^2 - x_3^2 + 10\,x_1x_3 = 1$

find the two points closest to the origin.

Consider $q(x_1,x_2,x_3) = \underbrace{-x_1^2 + x_2^2 - x_3^2 + 10\,x_1x_3}_{\text{two sheet hyperboloid?}}$, this has associated matrix:

$$A = \begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & -1 \end{bmatrix} \qquad q(\vec{x}) = c_1^2\lambda_1 + c_2^2\lambda_2 + c_3^2\lambda_3$$

$$f_A(\lambda) = \det(A - \lambda I_3) = \det\begin{bmatrix} -1-\lambda & 0 & 5 \\ 0 & 1-\lambda & 0 \\ 5 & 0 & -1-\lambda \end{bmatrix}$$

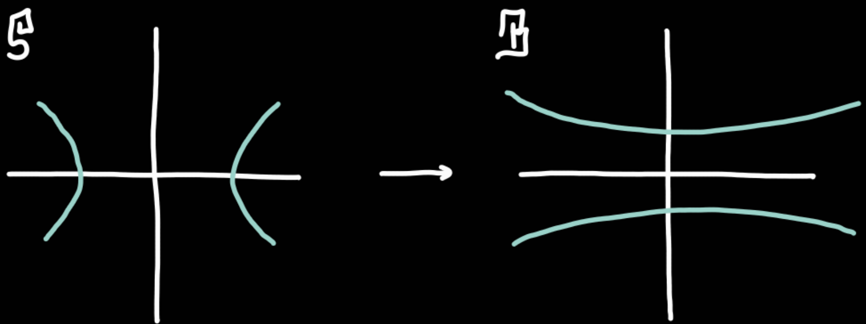
$$\lambda_1 = 1,\ \lambda_2 = -6,\ \lambda_3 = 4 \quad , \quad \mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

Thus $q(\vec{x}) = c_1^2 - 6c_2^2 + 4c_3^2$. Since we want to solve $q(\vec{x}) = 1$, we have:

$$c_1^2 - 6c_2^2 + 4c_3^2 = 1.$$

$c_1 = 0$, $c_2 = 0$ $c_3 = \frac{1}{2}$ $\quad [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$ has length $\frac{1}{2}$.

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$



1. Basis and notation

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} x_1 \\ \vdots \end{bmatrix} = \vec{x} = c_1 \cdot \vec{v}_1 + \cdots + c_m \cdot \vec{v}_m = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \end{bmatrix} =$$

$$\begin{bmatrix} c_m \end{bmatrix} \qquad \begin{bmatrix} x_n \end{bmatrix} \qquad \underbrace{\begin{bmatrix} | & & | \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} c_m \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}}$$

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$ $\qquad = S \cdot [\vec{x}]_{\mathcal{B}}$

Practice Final 7: $\underbrace{x_1 + 2x_2 + x_3 = 0}_{\text{gives a restriction for the coordinates of the vectors in } \mathcal{B}.}$, find $\mathcal{B}$ such that $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{gives a relation between the vectors in } \mathcal{B}.}$.

Practice Final 13: $\begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$ has three different real eigenvalues.

What are the signs? $\qquad a, b, c, d > 0$

Is the largest one positive or negative?

$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 \quad , \quad \mathrm{tr}(A) = 0$

$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \quad , \quad \det(A) = bcd > 0$

$- \; - \; +$

$- \; + \; -$ $\qquad$ Two of them will be negative, one positive.

$+ \; - \; -$

The positive one is the largest.

A quick check using a, b, c, d with 1 and 0 we can find all options.

Practice Final 10:

The least-squares solutions are solutions of the normal equation.

$A^T A \vec{x} = A^T \vec{b}$ $\longleftarrow$ solve this.