Topics:

1. How to do rref.

2. Solutions of a matrix in terms of rref and rank.

3. Invertibility of a matrix and computing inverses.

4. Finding kernels and images, interpreting geometrically.

5. Linear transformations geometrically.

6. How to see a linear transformation as a matrix.

7. Linear independence, kernels, and invertibility of matrices.

1. How to do rref.

$$\begin{bmatrix} 6 & 15 & 7 \\ 7 & 4 & 7 \\ 12 & 8 & 9 \end{bmatrix}$$

$\downarrow$ $R_1 \cdot \frac{1}{6}$

$$\begin{bmatrix} 1 & 5/2 & 7/6 \\ 7 & 4 & 7 \\ 12 & 8 & 9 \end{bmatrix}$$

$\downarrow$ $R_2 - 7 \cdot R_1$ $\quad R_3 - 12 \cdot R_1$

$$\begin{bmatrix} 1 & 5/2 & 7/6 \\ 0 & -27/2 & -7/6 \\ 0 & -22 & -5 \end{bmatrix}$$

$\downarrow$ $R_2 \cdot \frac{-2}{27}$

$$\begin{bmatrix} 1 & \frac{5}{2} & 7/6 \end{bmatrix}$$

3. Invertibility of a matrix and computing inverses.

$$\left[\begin{array}{ccc|ccc} 6 & 15 & 7 & 1 & 0 & 0 \\ 7 & 4 & 7 & 0 & 1 & 0 \\ 12 & 8 & 9 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc} \frac{1}{6} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc} \frac{1}{6} & 0 & 0 \\ -\frac{7}{6} & 1 & 0 \\ -2 & 0 & 1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc} \frac{1}{6} & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & \frac{7}{81} \\ 0 & -22 & -5 \end{bmatrix} \qquad \begin{bmatrix} \frac{7}{81} & \frac{-2}{27} & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$\downarrow R_1 - \frac{5}{2} R_2 \quad R_3 + 22 R_2 \downarrow$

$$\begin{bmatrix} 1 & 0 & \frac{77}{81} \\ 0 & 1 & \frac{7}{81} \\ 0 & 0 & -\frac{251}{81} \end{bmatrix} \qquad \begin{bmatrix} \frac{-8}{162} & \frac{5}{27} & 0 \\ \frac{7}{81} & \frac{-2}{27} & 0 \\ \frac{-8}{81} & \frac{-44}{27} & 1 \end{bmatrix}$$

$\downarrow R_3 \cdot \frac{-81}{251} \quad \downarrow$

$$\begin{bmatrix} 1 & 0 & \frac{77}{81} \\ 0 & 1 & \frac{7}{81} \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow R_2 - \frac{7}{81} \cdot R_3 \quad R_1 - \frac{77}{81} R_3 \downarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{20}{251} & -\frac{79}{251} & \frac{77}{251} \\ 0 & 1 & 0 & \frac{21}{251} & -\frac{30}{251} & \frac{7}{251} \\ 0 & 0 & 1 & \frac{8}{251} & \frac{132}{251} & -\frac{81}{251} \end{array}\right]$$

$$\underbrace{\begin{bmatrix} -\frac{20}{251} & -\frac{79}{251} & \frac{77}{251} \\ \frac{21}{251} & -\frac{30}{251} & \frac{7}{251} \\ \frac{8}{251} & \frac{132}{251} & -\frac{81}{251} \end{bmatrix}}_{\text{inverse}}$$

Q: How many solutions are there to the equation $A\vec{x} = \vec{b}$?

1 solution.

2. Solutions of a matrix in terms of rref and rank.

$[A \mid \vec{b}]$ augmented matrix of $A\vec{x} = \vec{b}$.

How about $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$? $\quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

equation $\rightsquigarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right] \longleftrightarrow \begin{aligned} 1\cdot x + 0\cdot y + 0\cdot z &= b_1 \\ 0\cdot x + 1\cdot y + 0\cdot z &= b_2 \\ 0\cdot x + 0\cdot y + 0\cdot z &= b_3 \end{aligned}$$

$\uparrow \ \uparrow \ \uparrow$

variable

Q: Are the columns of $\begin{bmatrix} 6 & 15 & 7 \end{bmatrix}$ linearly independent?

$$\begin{bmatrix} 7 & 4 & 7 \\ 12 & 8 & 9 \end{bmatrix}$$

The vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent if there are no real numbers a, b, c with $a\cdot\vec{v}_1 + b\cdot\vec{v}_2 + c\cdot\vec{v}_3 = 0$.

Equivalently none of the $\vec{v}_i$ is a linear combination of the others.

$$(-c_i)\vec{v}_i = c_1\cdot\vec{v}_1 + c_2\cdot\vec{v}_2 + \cdots + c_{i-1}\cdot\vec{v}_{i-1} + c_{i+1}\vec{v}_{i+1} + \cdots + c_m\cdot\vec{v}_m$$

$$0 = c_1\cdot\vec{v}_1 + \cdots + c_m\cdot\vec{v}_m$$

We can rewrite $a\vec{v}_1 + b\cdot\vec{v}_2 + c\cdot\vec{v}_3 = 0$ as $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\text{in the kernel of } \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}.} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$a \neq 0 \quad b \neq 0 \quad c \neq 0$

(at least two are not zero)

A linear dependence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ gives an element $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in the kernel of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$.

Computing the kernel of $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ is solving $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}}$.

So we are solving $A\vec{x} = \vec{0}$. This has only one solution (see previous question).

But $\vec{0} = \vec{x}$ is a solution! The only one!

So there are no other solutions to $a\cdot\vec{v}_1 + b\cdot\vec{v}_2 + c\cdot\vec{v}_3 = 0$

except $a=b=c=0$.

Sub-example: $\begin{bmatrix} 6 & 15 \\ 7 & 4 \\ 12 & 8 \end{bmatrix} \xrightarrow{\text{rref}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$ $\qquad \begin{bmatrix} 6 & 15 \\ 7 & 4 \\ 12 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

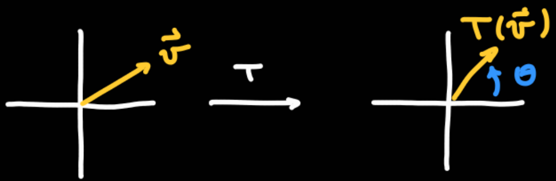
since every column has a leading one, the two original vectors are linearly independent.

$$\begin{array}{cc} & \text{variable} \\ \text{equation} \rightsquigarrow & \\ \rightsquigarrow & \end{array} \begin{bmatrix} 1 & 5 & 10 \\ 2 & 4 & 8 \end{bmatrix}$$

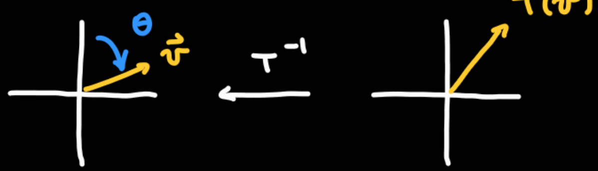
If we have more columns than rows, the columns are automatically dependent: we will always have at least one free variable.

5. Linear transformations geometrically.

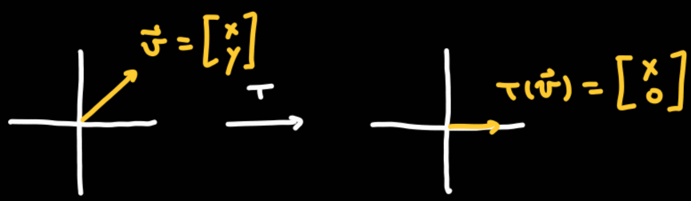
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$



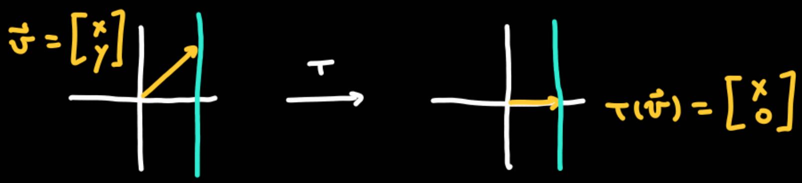
Note θ and $-\theta$ add up to 0.



$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



This is a projection onto the x-axis. Pick L a line perpendicular to the x-axis. Every vector $\vec v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that the point $(v_1, v_2) \in \mathbb{R}^2$ is in L is projected onto the same image $T(\vec v) = \begin{bmatrix} v_1 \\ \end{bmatrix}$



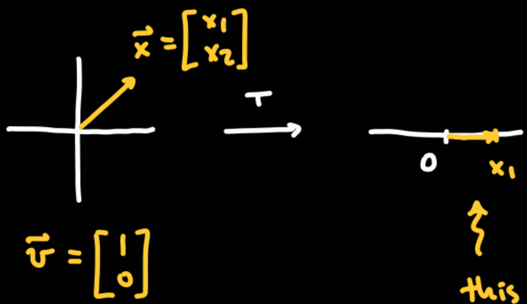
$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\ker(T) = \underbrace{\mathrm{span}(\vec{e}_2)}$

the set of linear combinations of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Pick $\vec{v} \in \mathbb{R}^2$ a non-zero vector. Then $T(\vec{x}) = \vec{x} \cdot \vec{v}$ is a linear transformation

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y$



this is a vector in one dimension

Computing the kernel of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: this is solving $A\vec{x} = \vec{0}$, so:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow$$

$x = 0$

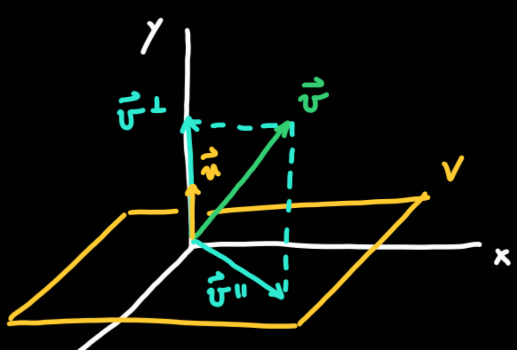
this has $y = t$ a free variable

The solutions are: $\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, all linear combinations of $\vec{e}_2$.

$$\mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

$$\mathrm{span}\underbrace{(v_1, v_2, v_3)}_{\text{if these are linearly independent}} \neq \mathrm{span}(w_1, w_2).$$

4. Finding kernels and images, interpreting geometrically.



Linear transformation: projection onto a plane.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\vec{v}^{\perp} = (\vec{v}\cdot\vec{u})\,\vec{u}$, $\vec{v}^{\parallel} = \vec{v} - \vec{v}^{\perp} = \text{proj}_V(\vec{v})$

$\begin{bmatrix} \text{proj}_V(\vec{e}_1) & \text{proj}_V(\vec{e}_2) & \text{proj}_V(\vec{e}_3) \end{bmatrix} \longleftarrow$ matrix of the linear transformation

$V = \text{im}(T) = \text{span}(\vec{e}_1, \vec{e}_3)$ it's the plane spanned by $\vec{e}_1$ and $\vec{e}_3$.

$\ker(T) = \text{span}(\vec{e}_2)$ it's the line spanned by the perpendicular vector.

6. How to see a linear transformation as a matrix.

$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}}_{A}$ is an $n \times m$ matrix.

since there are n rows in A , then $\vec{y}$ has
since the columns of A have n entries n entries

We look at $\vec{y} = A\vec{x}$.

since there are m columns in A , then $\vec{x}$ has
since the rows of A have m entries m entries

columns rows

$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$