To do:

1. Mathemagic.

2. Gram-Schmidt and QR factorization.

3. Orthogonal projections and least-squares solutions.

4. Basis of the image and kernel of a linear transformation, geometric interpretation.

5. Coordinates and change of basis.

1. Mathemagic.

$A\vec{x} = \vec{b}$ $\quad\quad$ $\vec{x}^*$

If $\ker(A) = \{\vec{0}\}$ then $\vec{x}^*$ is unique and $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$.

$\ker(A) = \{\vec{0}\} \longrightarrow \ker(A^TA) = \{\vec{0}\} \longrightarrow A^TA$ has an inverse

$A^TA\vec{x} = A^T\vec{b} \rightsquigarrow \vec{x} = (A^TA)^{-1}A^T\vec{b}$

A solution of $A^TA\vec{x} = A^T\vec{b}$ is also a solution of $A\vec{x} = \vec{b}$.

A solution of $A\vec{x} = \vec{b}$ is also a solution of $A^TA\vec{x} = A^T\vec{b}$.

5. Coordinates and change of basis.

$\vec{x}$ in $\mathbb{R}^n$ in $\mathfrak{B} = \{\vec{e}_1, \dots, \vec{e}_n\}$.

$\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \longleftrightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdots + c_n \vec{v}_n$$

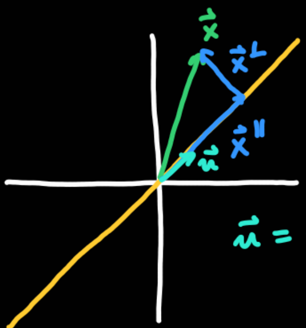
$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad \updownarrow \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = S \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\mathbb{R}^n, \mathcal{B} \xrightarrow{S} \mathbb{R}^n, \mathcal{S}$$

3. Orthogonal projections and least-squares solutions.

V subspace, $\vec{x}$ vector, we can decompose $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ where $\vec{x}^{\parallel}$ is in V and $\vec{x}^{\perp}$ is perpendicular to V (so $\vec{x}^{\perp}$ is in $V^{\perp}$).

$n=2$:



$L = \text{span}(\vec{v})$

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$

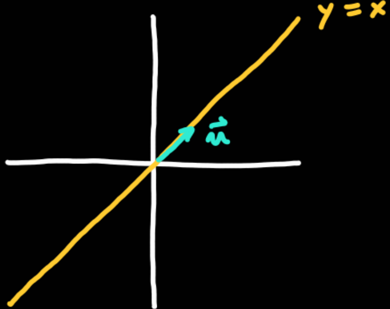
L is $y = mx$ so $L = \text{span}\left(\begin{bmatrix} 1 \\ m \end{bmatrix}\right)$

except if L is $x = 0$, where

$L = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$$



$L = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$$\vec{u} = \frac{1}{\sqrt{1+1}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The matrix associated to this projection is:

$$A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$$

$$T(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_2) = (\vec{e}_2 \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\vec{x}^{\parallel}$

$$K(\vec{e}_1) = \vec{e}_1 - (\vec{e}_1 \cdot \vec{u})\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$K(\vec{e}_2) = \vec{e}_2 - (\vec{e}_2 \cdot \vec{u})\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so $B = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\vec{e}_1 = \vec{e}_1^{\parallel} + \vec{e}_1^{\perp} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

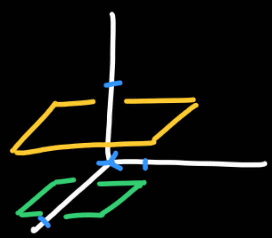
$$\vec{e}_2 = \vec{e}_2^{\parallel} + \vec{e}_2^{\perp} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

is the matrix associated to taking the orthogonal complement $\vec{x}^{\perp}$.

$n=3$:

Projection onto a line L:

$$L = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$$



$$\vec{u} = \frac{1}{\sqrt{1+4+9}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

To find the matrix associated to the projection into L we find:

$$T(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u})\vec{u} = \frac{1}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(\vec{e}_2) = (\vec{e}_2 \cdot \vec{u})\vec{u} = \frac{1}{7} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(\vec{e}_3) = (\vec{e}_3 \cdot \vec{u})\vec{u} = \frac{3}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

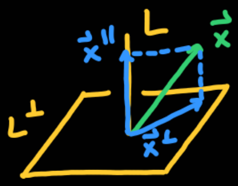
$$\begin{bmatrix} \frac{1}{14} & \frac{1}{7} & \frac{3}{14} \\ \frac{1}{7} & \frac{2}{7} & \frac{3}{7} \\ \frac{3}{14} & \frac{3}{7} & \frac{9}{14} \end{bmatrix}$$

$$K(\vec{e}_1) = \vec{e}_1 - (\vec{e}_1 \cdot \vec{u})\vec{u} = \begin{bmatrix} 13/14 \\ -2/14 \\ -3/14 \end{bmatrix}$$

$$K(\vec{e}_2) = \vec{e}_2 - (\vec{e}_2 \cdot \vec{u})\vec{u} = \begin{bmatrix} -1/7 \\ 5/7 \\ -3/7 \end{bmatrix}$$

$$K(\vec{e}_3) = \vec{e}_3 - (\vec{e}_3 \cdot \vec{u})\vec{u} = \begin{bmatrix} -3/14 \\ -6/14 \\ 5/14 \end{bmatrix}$$

Remark: When we decompose $\underbrace{\vec{x} = \vec{x}^{\parallel}}_{\text{in } L} + \underbrace{\vec{x}^{\perp}}_{\text{in } L^{\perp}}$, choosing $\vec{x}^{\perp}$ is the same as projecting onto $L^{\perp}$.



Projection onto a plane V:

$$V = L^{\perp} = \left(\operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)\right)^{\perp} \qquad x + 2y + 3z = 0$$

We want a basis of V, so we find two vectors in V that are linearly independent.

$$\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1}, \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$$ is a basis, we now transform it into an orthonormal basis. We use Gram-Schmidt:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{4+1}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\sqrt{\frac{9}{25} + \frac{36}{25} + 1}}\begin{bmatrix} -3/5 \\ -6/5 \\ 1 \end{bmatrix} = \frac{5}{\sqrt{70}}\begin{bmatrix} -3/5 \\ -6/5 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{70}}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$$

$$\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}, \quad \vec{v}_2^{\parallel} = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \frac{6}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \begin{bmatrix} -3/5 \\ -6/5 \\ 1 \end{bmatrix}$$

$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, $\vec{x}^{\parallel}$ in V, $\vec{x}^{\perp}$ in $V^{\perp}$ then:

$$\operatorname{proj}_V(\vec{x}) = (\vec{x} \cdot \vec{u}_1)\vec{u}_1 + (\vec{x} \cdot \vec{u}_2)\vec{u}_2$$

$$T(\vec{e}_1) = (\vec{e}_1 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_1 \cdot \vec{u}_2)\vec{u}_2 = -\frac{2}{5}\begin{bmatrix} -2 \\ 1 \\ \end{bmatrix} + \frac{-3}{70}\begin{bmatrix} -3 \\ -6 \\ \end{bmatrix} = \begin{bmatrix} 65/70 \\ \end{bmatrix} = \begin{bmatrix} 13/14 \\ -1 \end{bmatrix}$$

$$\ldots \begin{bmatrix} \vdots \\ 0 \end{bmatrix} \ldots \begin{bmatrix} \vdots \\ 5 \end{bmatrix} \ldots \begin{bmatrix} -10/70 \\ -15/70 \end{bmatrix} \ldots \begin{bmatrix} \ldots/7 \\ -3/14 \end{bmatrix}$$

$$T(\vec{e}_2) = (\vec{e}_2 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_2 \cdot \vec{u}_2)\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{-6}{70}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -10/70 \\ 50/70 \\ -30/70 \end{bmatrix} = \begin{bmatrix} -1/7 \\ 5/7 \\ -3/7 \end{bmatrix}$$

$$T(\vec{e}_3) = (\vec{e}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{e}_3 \cdot \vec{u}_2)\vec{u}_2 = 0 \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{70}\begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -15/70 \\ -30/70 \\ 25/70 \end{bmatrix} = \begin{bmatrix} -3/14 \\ -3/7 \\ 5/14 \end{bmatrix}$$

6. Sketch least-squares solutions.

$A\vec{x} = \vec{b}$ $\quad$ A is $n \times m$

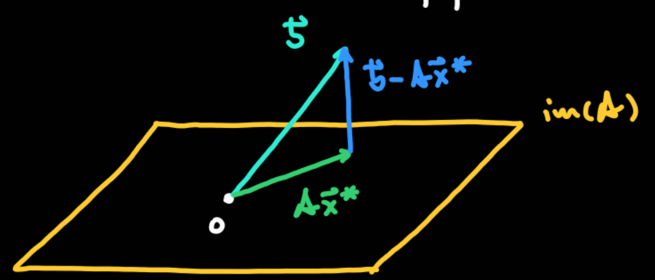
$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$$

$\mathrm{im}(A)$ is a plane, $\vec{b}$ is some vector not in $\mathrm{im}(A)$, $A\vec{x}^*$ is in $\mathrm{im}(A)$

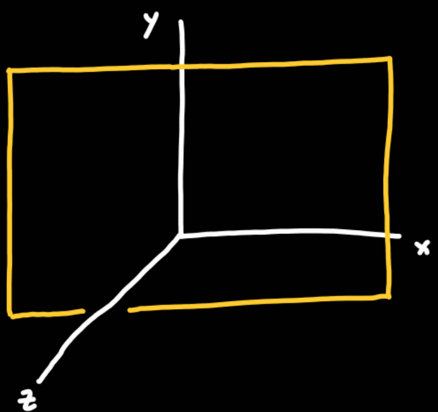
projection of $\vec{b}$ in $\mathrm{im}(A)$

$\vec{b} - A\vec{x}^*$ is in $(\mathrm{im}(A))^\perp$

$\vec{b} - A\vec{x}^*$ is perpendicular to $\mathrm{im}(A)$



$\mathrm{im}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$ is the plane $z=0$



$\vec{b} - A\vec{x} = \vec{0}$

2. Gram-Schmidt and QR factorization.

Let $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$, then:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} \quad \text{where } \vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp} \text{ with respect to } \operatorname{span}(\vec{v}_1)$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} \quad \text{where } \vec{v}_3 = \vec{v}_3^{\parallel} + \vec{v}_3^{\perp} \text{ with respect to } \operatorname{span}(\vec{v}_1, \vec{v}_2)$$

$$\vdots$$

$$\vec{u}_n = \frac{\vec{v}_n^{\perp}}{\|\vec{v}_n^{\perp}\|} \quad \text{where } \vec{v}_n = \vec{v}_n^{\parallel} + \vec{v}_n^{\perp} \text{ with respect to } \operatorname{span}(\vec{v}_1, \dots, \vec{v}_{n-1})$$

Setting $M = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ we can decompose:

$$M = QR = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix}}_{\text{orthogonal}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix}}_{\text{upper triangular}}$$

with $r_{11} = \|\vec{v}_1\|$, $r_{22} = \|\vec{v}_2^{\perp}\|, \dots,$ $r_{nn} = \|\vec{v}_n^{\perp}\|$.

$$r_{ij} = \vec{u}_i \cdot \vec{v}_j = \vec{u}_i \cdot (c_1 \cdot \vec{u}_1 + \dots + c_{j-1} \cdot \vec{u}_{j-1} + \vec{v}_j^{\perp}) = \vec{u}_i \cdot \vec{v}_j^{\perp}$$

$$\vec{v}_j = \underbrace{\vec{v}_j^{\parallel}}_{} + \vec{v}_j^{\perp}$$

$$\operatorname{span}(\vec{v}_1, \dots, \vec{v}_{j-1}) = \operatorname{span}(\vec{u}_1, \dots, \vec{u}_{j-1})$$

$$\vec{v}_j = c_1 \cdot \vec{u}_1 + \dots + c_{j-1} \vec{u}_{j-1} + \vec{v}_j^{\perp}$$

Problem 3 Practice Midterm 2:

$$M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ \cdots \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 5 \\ \cdots \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}}_{Q'} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{R'}$$

1. Multiply, do Gram-Schmidt, compute R.

2. Transform Q' and R' into legitimate Q and R.