To do:

1. Basis of the image, basis of the kernel, geometric interpretation.

2. Least squares solution.

1. Basis of the image, basis of the kernel, geometric interpretation.

$$A, \ 3\times 4, \quad A : \mathbb{R}^4 \longrightarrow \mathbb{R}^3, \quad A = \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ \hline 3 & 1 & 8 & 4 \\ 5 & 3 & 4 & 9 \\ 6 & 2 & 0 & 7 \end{array} \quad \begin{matrix} 4 \\ 9 \\ 7 \end{matrix}$$

$$\begin{bmatrix} 3 & 1 & 8 & 4 \\ 5 & 3 & 4 & 9 \\ 6 & 2 & 0 & 7 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 1/3 & 8/3 & 4/3 \\ 5 & 3 & 4 & 9 \\ 6 & 2 & 0 & 7 \end{bmatrix} \xrightarrow[R_3-6R_1]{R_2-5R_1} \begin{bmatrix} 1 & 1/3 & 8/3 & 4/3 \\ 0 & 4/3 & -28/3 & 7/3 \\ 0 & 0 & -48/3 & -3/3 \end{bmatrix}$$

$$\xrightarrow{\frac{3}{4}\cdot R_2} \begin{bmatrix} 1 & 1/3 & 8/3 & 4/3 \\ 0 & 1 & -7 & 7/4 \\ 0 & 0 & -16 & -1 \end{bmatrix} \xrightarrow[\frac{1}{-16}\cdot R_3]{R_1-\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 & 15/3 & 9/12 \\ 0 & 1 & -7 & 7/4 \\ 0 & 0 & 1 & 1/16 \end{bmatrix}$$

$$\xrightarrow[R_2+7R_3]{R_1-\frac{15}{3}R_3} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \\ \hline 1 & 0 & 0 & 7/16 \\ 0 & 1 & 0 & 35/16 \\ 0 & 0 & 1 & \underline{1/16} \end{array} \quad \begin{matrix} 7/16 \\ 35/16 \\ 1/16 \end{matrix} \qquad \begin{bmatrix} 7/16 \\ 35/16 \\ 1/16 \\ 0 \\ -1 \end{bmatrix}$$

$\operatorname{im}(A)$:

$$\begin{bmatrix} 7/16 \\ 35/16 \\ 1/16 \\ -1 \\ 0 \end{bmatrix}$$

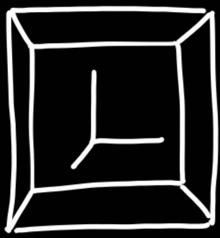
It is the span of the columns of A. By the rref(A) we know that the first three columns are linearly independent. We also know that the last column is a linear combination of the others.

$$\operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} \right)$$

Geometrically, these are three vectors in $\mathbb{R}^3$, so $\operatorname{im}(A) = \mathbb{R}^3$.

$A : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$

Equivalently $\text{im}(A)$ is a 3D space inside $\mathbb{R}^3$.



← attempt to put $\mathbb{R}^3$ inside $\mathbb{R}^4$

$\ker(A)$:

We know that the last column is a linear combination of the others, so

$\ker(A)$ has dimension 1.

$$\overset{\vec{v}_4}{\begin{bmatrix}4\\9\\7\end{bmatrix}} = \frac{7}{16}\cdot\overset{\vec{v}_1}{\begin{bmatrix}3\\5\\6\end{bmatrix}} + \frac{35}{16}\cdot\overset{\vec{v}_2}{\begin{bmatrix}1\\3\\2\end{bmatrix}} + \frac{1}{16}\overset{\vec{v}_3}{\begin{bmatrix}8\\4\\0\end{bmatrix}}$$

$$= \begin{bmatrix} 21/16 + \frac{35}{16} + \frac{8}{16} \\ 35/16 + \frac{105}{16} + \frac{4}{16} \\ 42/16 + \frac{70}{16} + 0 \end{bmatrix} = \begin{bmatrix} 64/16 \\ \frac{144}{16} \\ \frac{112}{16} \end{bmatrix} = \begin{bmatrix}4\\9\\7\end{bmatrix}$$

$$\frac{7}{16}\cdot\overset{\vec{v}_1}{\begin{bmatrix}3\\5\\6\end{bmatrix}} + \frac{35}{16}\cdot\overset{\vec{v}_2}{\begin{bmatrix}1\\3\\2\end{bmatrix}} + \frac{1}{16}\overset{\vec{v}_3}{\begin{bmatrix}8\\4\\0\end{bmatrix}} - \overset{\vec{v}_4}{\begin{bmatrix}4\\9\\7\end{bmatrix}} = 0$$

$$\begin{array}{c} \begin{array}{cccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{array} \\ \begin{bmatrix} 3 & 1 & 8 & 4 \\ 5 & 3 & 4 & 9 \\ 6 & 2 & 0 & 7 \end{bmatrix} \end{array} \underbrace{\begin{bmatrix} 7/16 \\ 35/16 \\ 1/16 \\ -1 \end{bmatrix}}_{\text{in } \ker(A)} = 0$$

So $\begin{bmatrix} 7/16 \\ 35/16 \\ 1/16 \\ -1 \end{bmatrix}$ is in $\ker(A)$, which has dimension 1, so $\ker(A) = \text{span}\left(\begin{bmatrix} 7/16 \\ 35/16 \\ 1/16 \\ -1 \end{bmatrix}\right)$.

Geometrically, this is one line in $\mathbb{R}^4$.

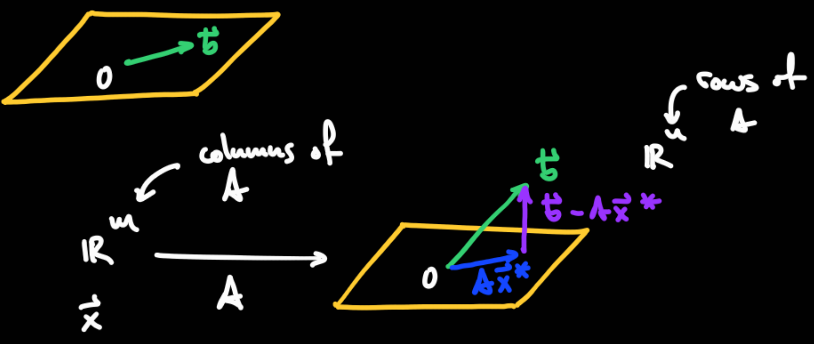
2. Least squares solution.

$A = \begin{bmatrix} 2 & 6 \\ 5 & 15 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, find least-squares solution(s) of $A\vec{x} = \vec{b}$.

$$A\vec{x} = \text{proj}_{\text{im}(A)}(\vec{b})$$

1. $\vec{b}$ is in $\text{im}(A)$

2. $\vec{b}$ is <u>not</u> in $\text{im}(A)$



$A\vec{x}^*$ is $\text{proj}_{\text{im}(A)}(\vec{b})$

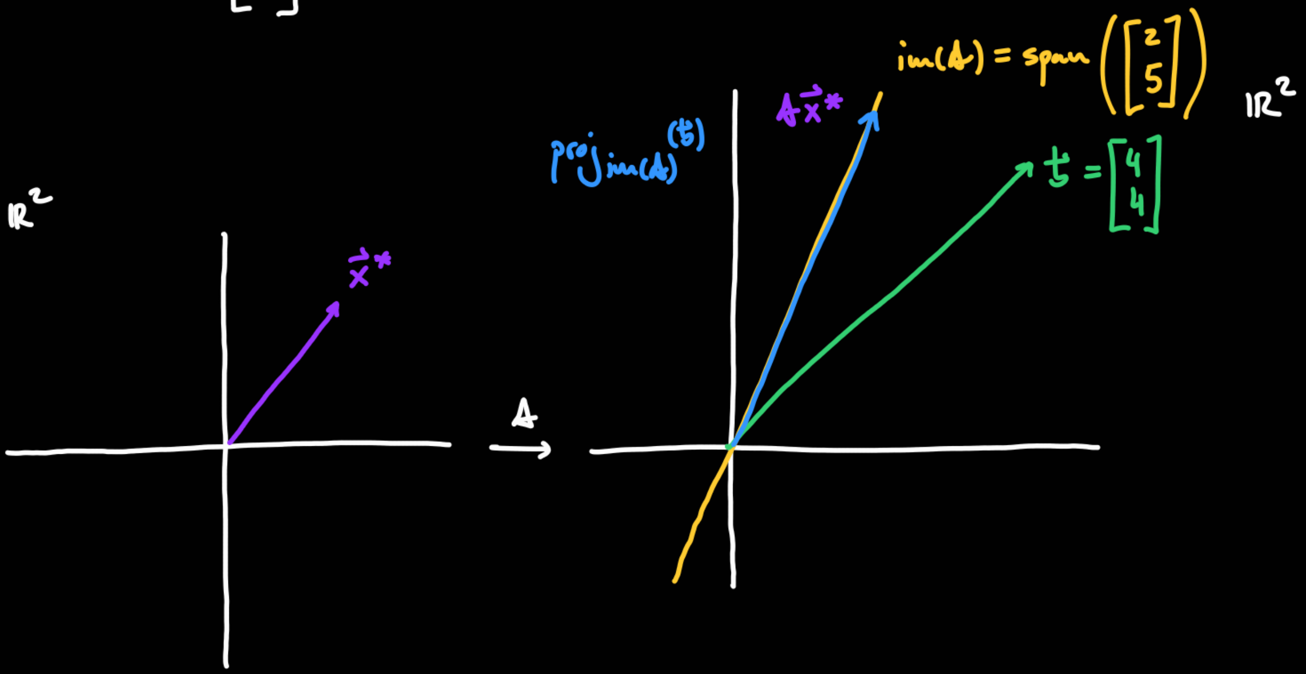
Note that $\ker(A) \neq \{\vec{0}\}$, so we <u>cannot</u> use that

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}.$$

We know that $\text{im}(A) = \text{span}\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$, we compute:

$$\text{proj}_{\text{im}(A)}(\vec{b}) = (\vec{b}\cdot\vec{u})\vec{u} = \frac{1}{29}\cdot 28 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\vec{u} = \frac{1}{\sqrt{29}}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$



We want to solve:

$$\left[\begin{array}{cc|c} 2 & 6 & \frac{28}{29}\cdot 2 \\ 5 & 15 & \frac{28}{29}\cdot 5 \end{array}\right] \xrightarrow{\frac{1}{2}\cdot R_1} \left[\begin{array}{cc|c} 1 & 3 & \frac{28}{29} \\ 5 & 15 & \frac{28\cdot 5}{29} \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cc|c} 1 & 3 & \frac{28}{29} \\ 0 & 0 & 0 \end{array}\right]$$

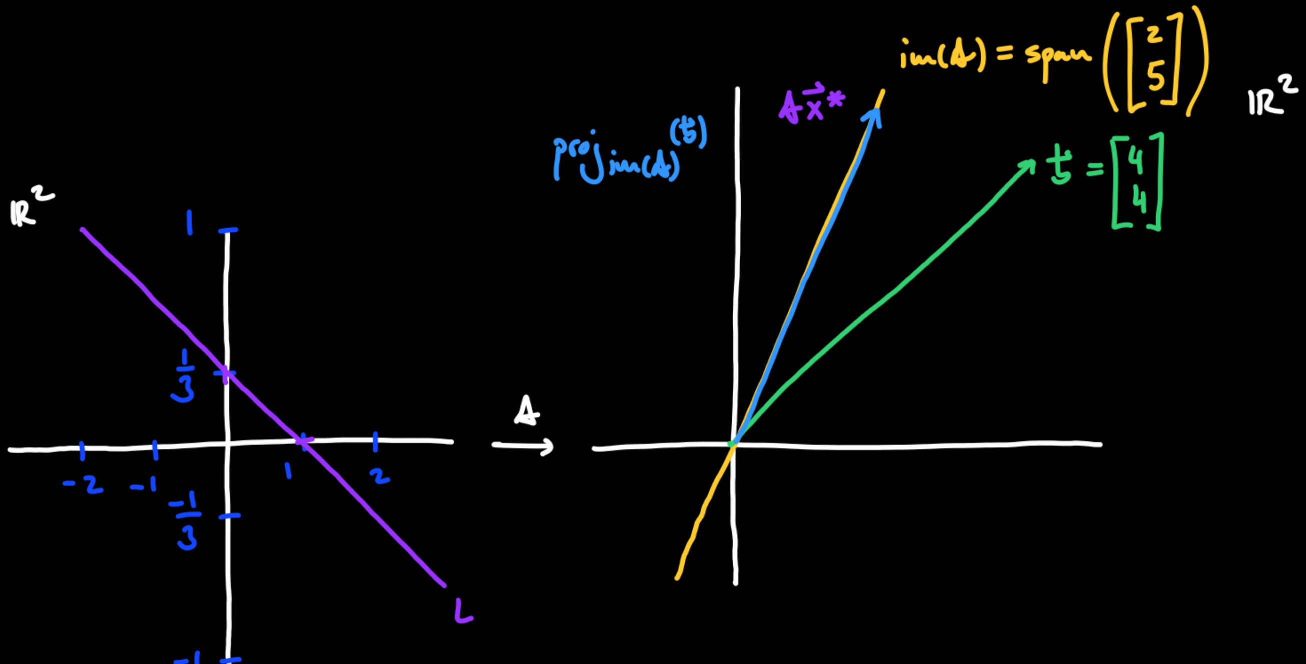
$$\underbrace{\begin{matrix} 2 & 6 \\ 5 & 15 \end{matrix}}_{A} \qquad \underbrace{\begin{matrix} \frac{28}{29}\cdot 2 \\ \frac{28}{29}\cdot 5 \end{matrix}}_{\text{proj}_{\text{im}(A)}(\vec{b})}$$

$$x + 3y = \frac{28}{29}$$

Set $y = t$ free variable then:

$$\vec{x}^* = \begin{bmatrix} \frac{28}{29} - 3t \\ t \end{bmatrix}$$

$$L: \quad y = \frac{-x}{3} + \frac{28}{87}$$



$$y = \frac{-x}{3} + \frac{28}{87}$$

If A is invertible, we always have a unique solution:

1. $\vec{b}$ is in $\text{im}(A)$ $\quad \vec{x}^* = A^{-1}\vec{b}$.

2. $\vec{b}$ is <u>not</u> in $\text{im}(A)$ $\quad \vec{x}^* = A^{-1}\,\text{proj}_{\text{im}(A)}(\vec{b})$.

Problem 3 Practice Midterm 2:

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$ find the QR factorization of M.

$$\underbrace{\qquad\qquad}_{Q'} \quad \underbrace{\qquad}_{R'}$$

$M = QR$, Q orthogonal and R upper triangular with strictly positive diagonal entries.

M has size 4×2, so Q has size 4×2 and R has size 2×2.

The last two columns of Q' are completely unnecessary. The bottom two rows of R' are completely superfluous.

In fact $M = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}}_{R}$ is already the QR decomposition of M.