Problem 5.3.35:

Find an orthogonal transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$T = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \qquad \vec{u}_1 = T(\vec{e}_1), \ \vec{u}_2 = T(\vec{e}_2), \ \vec{u}_3 = T(\vec{e}_3)$$

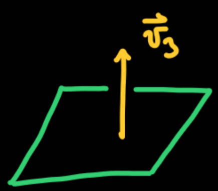
Orthogonal transformations always have inverses, and they are computed via transposing.

$$T^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = T^{-1}T\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \qquad \text{so } T^{-1}(\vec{e}_3) = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

If we find two additional vectors that are orthogonal and orthogonal to $\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ and have length one, we are done.

$$T = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \qquad T\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$



$$\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z = 0 \qquad 2x + 2y + z = 0$$

$$2 \cdot \frac{-2}{3} + 2 \cdot \frac{1}{3} + \frac{2}{3} = 0$$

$$\vec{v}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\vec{v}_3 \cdot \vec{v}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \frac{2}{3} \cdot \frac{-2}{3} + \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = 0$$

Using that $\vec{v}_2$ has to be perpendicular to $\vec{v}_1$ and $\vec{v}_3$, and in V, we can find:

$$\vec{v}_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \qquad \text{(or compute the cross product)}$$

Now:

$$T^{-1} = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$, and taking inverses and transposes of orthogonal matrices is the same thing, so:

$$T = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ -1/3 & 2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

<u>orthogonal matrices</u>: square matrices whose columns form an orthonormal basis.

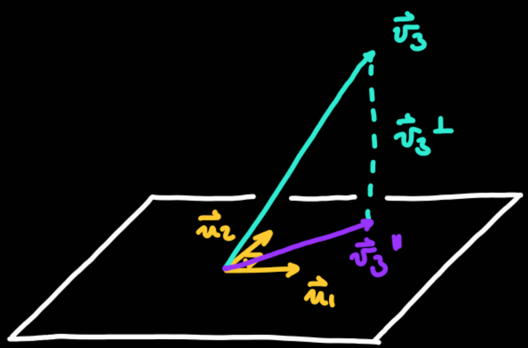
$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i\cdot 0 - j\cdot 0 + k\cdot 1 = k\cdot 1$$

We want $\begin{array}{c} \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} x & y & z \end{matrix} \end{array}$ to have <u>non</u>-zero determinant.

<u>Problem 5.2.13.</u>:

Gram-Schmidt : $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \underset{\vec{v}_3}{\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}}$

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

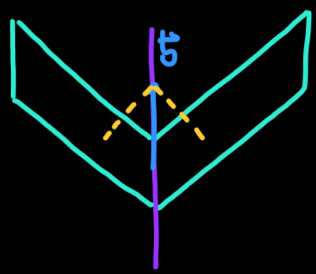


$$\vec{v}_3'' = (\vec{v}_3\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}_3\cdot\vec{u}_2)\vec{u}_2 = \left(\tfrac{2}{2} + \tfrac{1}{2} - \tfrac{1}{2}\right)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \left(\tfrac{-2}{2} - \tfrac{1}{2} - \tfrac{1}{2}\right)\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ \ldots \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ \ldots \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \\ 3/2 \\ \ldots \end{bmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 3/2 \\ 3/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \|\vec{v}_3^{\perp}\| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}} = 1$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$



Theorem 5.4.7: $A\vec{x} = \vec{b}$ $\quad A$ $n \times m$

Recall: If $\ker(A) = \{\vec{0}\}$ then $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$.

Why? Because if $\ker(A) = \{\vec{0}\}$ then A^TA is invertible, so the normal equation $A^TA\vec{x} = A^T\vec{b}$ has unique solution $\vec{x} = (A^TA)^{-1}A^T\vec{b}$.

Also, finding $\vec{x}^*$ is finding the projection of $\vec{b}$ onto $\text{im}(A)$. How?

Well, $\text{proj}_{\text{im}(A)}(\vec{b}) = A\vec{x}^*$.

Then: $\text{proj}_V(\vec{b}) = A(A^TA)^{-1}A^T\vec{b}$. $\qquad \text{proj}_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\quad \vec{b} \longmapsto A(A^TA)^{-1}A^T\vec{b}$

(as long as $\ker(A) = \{\vec{0}\}$)

Why is $\vec{x} = (A^TA)^{-1}A^T\vec{b}$ a least-squares solution of $A\vec{x} = \vec{b}$?

Why is a solution of $A^TA\vec{x} = A^T\vec{b}$ a least-squares solution of $A\vec{x} = \vec{b}$?

$$A^TA\vec{x}^* = A^T\vec{b} \longleftrightarrow A^T\vec{b} - A^TA\vec{x}^* = \vec{0} \longleftrightarrow A^T(\vec{b} - A\vec{x}^*) = \vec{0}$$

$$\longleftrightarrow \vec{b} - A\vec{x}^* \text{ is in } \ker(A^T) = (\text{im}(A))^{\perp} \longleftrightarrow A\vec{x}^* = \text{proj}_{\text{im}(A)}\vec{b}$$

clear conceptually, harder

Pythagoras Theorem

mathematically rigorously

Decompose:

$$\underbrace{\vec{b} - A\vec{x}^*}_{\text{im}(A)^\perp} = \underbrace{\vec{b}^{\parallel}}_{\text{im}(A)^\perp} + \underbrace{\vec{b}^{\perp} - A\vec{x}^*}_{\underbrace{\text{im}(A)}_{\text{must be } \vec{0}}}$$

$\longleftrightarrow \quad \|\vec{b} - A\vec{x}^*\| \leq \|\vec{b} - A\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^m$

$\longleftrightarrow \quad \vec{x}^*$ is a least-squares solution of $A\vec{x} = \vec{b}$.

Why do we need $\ker(A) = \{\vec{0}\}$?

Note that $\ker(A) = \ker(A^T A)$:

1. If $\vec{v}$ in $\ker(A)$ then $A^T A\vec{v} = A^T(A\vec{v}) = A^T\vec{0} = \vec{0}$ so $\vec{v}$ is in $\ker(A^T A)$.

2. If $\vec{v}$ in $\ker(A^T A)$ then $A^T A\vec{v} = \vec{0}$. Since $A\vec{v}$ is in $\text{im}(A)$ and $\ker(A^T)$ because $A^T(A\vec{v}) = \vec{0}$, then :

   $A\vec{v}$ is in $\text{im}(A)$ and $(\text{im}(A))^\perp$, so $A\vec{v} = \vec{0}$.

   So $\vec{v}$ is in $\ker(A)$.

If $\ker(A) = \{\vec{0}\}$ then $\ker(A^T A) = \{\vec{0}\}$, but $A^T A$ is a square matrix. So $A^T A$ is invertible.

$P = QQ^T$, why is P symmetric ? $\quad P^T = P$

$$P^T = (QQ^T)^T = (Q^T)^T Q^T = QQ^T = P$$