Math 33A
Linear Algebra and Applications

Discussion for January 24-28, 2022

Problem 1.

Consider a matrix A of the form

$$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix},$$

where $a^2 + b^2 = 1$ and $a \neq 1$. Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to the basis

$$\begin{bmatrix} b \\ 1-a \end{bmatrix}, \begin{bmatrix} a-1 \\ b \end{bmatrix}.$$

Interpret the answer geometrically.

Solution: There are two ways of seeing this, one more geometric, the other more algebraic. Geometrically, the vector $\vec{v_1} = \begin{bmatrix} b \\ 1-a \end{bmatrix}$ determines a line in $\mathbb{R}^2$, and the vector $\vec{v_2} = \begin{bmatrix} a-1 \\ b \end{bmatrix}$ is perpendicular to this line. The matrix A is representing a reflection about the line parallel to $\vec{v_1}$. In the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$ a reflection about this line keeps $\vec{v_1}$ untouched and changes the sign of $\vec{v_2}$, and thus a reflection about this line has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Algebraically, the matrix is given by applying the linear transformation to $\vec{v_1}$ and putting the result in the first column, and then applying the linear transformation to $\vec{v_2}$ and putting the result in the second column, giving

$$\begin{aligned}
\begin{bmatrix} [T(\vec{v_1})]_{\mathfrak{B}} & [T(\vec{v_2})]_{\mathfrak{B}} \end{bmatrix} &= \begin{bmatrix} \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} b \\ 1-a \end{bmatrix}\right]_{\mathfrak{B}} & \left[\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1-a \\ -b \end{bmatrix}\right]_{\mathfrak{B}} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} ab + b - ba \\ b^2 + a^2 - a \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} a^2 + b^2 - a \\ ba - b - ab \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} b \\ 1-a \end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix} 1-a \\ -b \end{bmatrix}_{\mathfrak{B}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\end{aligned}$$

Problem 2.

Let A and B be square matrices, if there is an invertible matrix S such that $B = S^{-1}AS$ we say that A is similar to B. Find an invertible 2×2 matrix S such that

$$S^{-1} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} S$$

is of the form

$$\begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}.$$

What can you say about two of those matrices?

Solution: Since S is a 2×2 matrix, it has four unknowns. Leaving b and d representing any two real numbers, we have the equation

$$\frac{1}{xw - yz} \begin{bmatrix} w & -y \\ -z & x \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & b \\ 1 & d \end{bmatrix}$$

which (interestingly enough, see Problem 4 for more details about this) forces $b = 2$ and $d = 5$. Setting x and w as free variables, these four equations impose the restrictions $2y = w - x$ and $4z = w - 3x$. Since S has to be invertible, we have the additional restriction $xw - yz = \det(S) \neq 0$, which with the above solutions becomes $w^2 - 12wx + 3x^2 \neq 0$. Thus, as long as this invertibility condition is satisfied, we have

$$S = \begin{bmatrix} x & \frac{w-x}{2} \\ \frac{w-3x}{4} & w \end{bmatrix}.$$

The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is similar to the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix}$.

Problem 3.

If A is a 2×2 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

show that A is similar to a diagonal matrix D. Find an invertible S such that $S^{-1}AS = D$.

Solution: Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by A, namely $T(\vec{x}) = A\vec{x}$. Since we are given the image of $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, consider the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$. The matrix of T with respect to $\mathfrak{B}$ is

$$D = \big[[T(\vec{v_1})]_{\mathfrak{B}} \quad [T(\vec{v_2})]_{\mathfrak{B}}\big] = \left[\begin{bmatrix} 3 \\ 6 \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} -2 \\ -1 \end{bmatrix}_{\mathfrak{B}}\right] = \left[S^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad S^{-1}\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since we have changed from the standard basis to a new basis, we have $A = SDS^{-1}$, and thus $D = S^{-1}AS$ so A is similar to D.

Problem 4.

If $c \neq 0$, find the matrix of the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x}$$

with respect to the basis

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}a\\c\end{bmatrix}.$$

Solution: Denote this basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$, the matrix of T with respect to $\mathfrak{B}$ is

$$\begin{aligned}
\begin{bmatrix}[T(\vec{v_1})]_{\mathfrak{B}} & [T(\vec{v_2})]_{\mathfrak{B}}\end{bmatrix} &= \begin{bmatrix}\left[\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right]_{\mathfrak{B}} & \left[\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix}\right]_{\mathfrak{B}}\end{bmatrix}\\
&= \begin{bmatrix}\begin{bmatrix}a\\c\end{bmatrix}_{\mathfrak{B}} & \begin{bmatrix}a^2+bc\\ac+cd\end{bmatrix}_{\mathfrak{B}}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & a\\0 & c\end{bmatrix}^{-1}\begin{bmatrix}a\\c\end{bmatrix} & \begin{bmatrix}1 & a\\0 & c\end{bmatrix}^{-1}\begin{bmatrix}a^2+bc\\ac+cd\end{bmatrix}\end{bmatrix}\\
&= \begin{bmatrix}\begin{bmatrix}1 & -a/c\\0 & 1/c\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix} & \begin{bmatrix}1 & -a/c\\0 & 1/c\end{bmatrix}\begin{bmatrix}a^2+bc\\ac+cd\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0 & bc-ad\\1 & a+d\end{bmatrix}.
\end{aligned}$$

This explains what is going on in Problem 2. Setting $\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$, via a change of basis it will be similar to a matrix of the form $\begin{bmatrix}0 & bc-ad\\1 & a+d\end{bmatrix} = \begin{bmatrix}0 & 2\\1 & 5\end{bmatrix}$, forcing the mysterious appearance of the column $\begin{bmatrix}2\\5\end{bmatrix}$. Moreover, this forces $\vec{v_2} = \begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$. In particular, using as basis the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}a\\c\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$, we have that $S = \begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}$ is a solution for Problem 2.

Problem 5.

Is there a basis $\mathfrak{B}$ of $\mathbb{R}^2$ such that $\mathfrak{B}$-matrix B of the linear transformation

$$T(\vec{x}) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\vec{x}$$

is upper triangular?

Solution: No. Note first that T is a rotation of angle $\pi/2$. Note second that if T could be written as an upper triangular matrix in the basis $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}\}$ that would mean that $T(\vec{v_1}) = k\vec{v_1}$ for some real scalar k. In other words, $T(\vec{v_1})$ would be parallel to $\vec{v_1}$. However, since T is a rotation, this is impossible.