Math 33A
Linear Algebra and Applications

Discussion for January 31-February 4, 2022

Problem 1.

Here is an infinite-dimensional version of Euclidean space: In the space of all infinite sequences, consider the subspace ℓ_2 of square-summable sequences (namely, those sequences $(x_1, x_2, \dots)$ for which the infinite series $x_1^2 + x_2^2 + \cdots$ converges). For x and y in ℓ_2, we define

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots} \quad \text{and} \quad \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots .$$

A preliminary question is, why do $||\vec{x}||$ and $\vec{x} \cdot \vec{y}$ make sense, that is, why are they finite real numbers?

(a) Check that $\vec{x} = (1, 1/2, 1/4, 1/8, 1/16, \dots)$ is in ℓ_2, and find $||\vec{x}||$. Recall the formula for the geometric series: $1 + a + a^2 + a^3 + \cdots = 1/(1-a)$ if $-1 < a < 1$.

(b) Find the angle between $(1, 0, 0, 0, \dots)$ and $(1, 1/2, 1/4, 1/8, \dots)$.

(c) Give an example of a sequence $(x_1, x_2, \dots)$ that converges to 0 ($\lim_{n\to\infty} x_n = 0$) but does not belong to ℓ_2.

(d) Let L be the subspace of ℓ_2 spanned by $(1, 1/2, 1/4, 1/8, \dots)$. Find the orthogonal projection of $(1, 0, 0, 0, \dots)$ onto L.

The Hilbert space ℓ_2 was initially used mostly in physics: Werner Heisenberg's formulation of quantum mechanics is in terms of ℓ_2. Today, this space is used in many other applications, including economics. See, for example, the work of the economist Andreu Mas-Colell of the University of Barcelona.

Problem 2.

Give an algebraic proof for the triangle inequality

$$||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||.$$

Draw a sketch.

Problem 3.

(a) Consider a vector $\vec{v}$ in $\mathbb{R}^n$, and a scalar k. Show that $||k\vec{v}|| = |k|||\vec{v}||$.

(b) Show that if $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$, then $\vec{u} = \frac{\vec{v}}{||\vec{v}||}$ is a unit vector.

Problem 4.

Can you find a line L in $\mathbb{R}^n$ and a vector $\vec{x}$ in $\mathbb{R}^n$ such that $\vec{x} \cdot \text{proj}_L \vec{x}$ is negative? Explain, arguing algebraically.