Math 33A
Linear Algebra and Applications

Discussion for February 21-25, 2022

Problem 1.

In his groundbreaking text Ars Magna, the Italian mathematician Gerolamo Cardano explains how to solve cubic equations. In Chapter XI, he considers the following example: $x^3 + 6x = 20$.

(a) Explain why this equation has exactly one (real) solution. Here, this solution is easy to find by inspection. The point of this exercise is to show a systematic way to find it.

(b) Cardano explains his method as follows (we are using modern notation for the variables): "I take two cubes v^3 and u^3 whose difference shall be 20, so that the product vu shall be 2, that is, a third of the coefficient of the unknown x. Then, I say that $v - u$ is the value of the unknown x". Show that if v and u are chosen as stated by Cardano, then $x = v - u$ is indeed the solution of the equation $x^3 + 6x = 20$.

(c) Solve the system

$$\begin{aligned} v^3 - u^3 &= 20 \\ vu &= 2 \end{aligned}$$

to find u and v.

(d) Consider the equation $x^3 + px = q$, where p is positive. Using your previous work as a guide, show that the unique solution of this equation is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Check that this solution can also be written as

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

What can go wrong when p is negative?

(e) Consider an arbitrary cubic equation $x^3 + ax^2 + bx + c = 0$. Show that the substitution $x = t - (a/3)$ allows you to write this equation as $t^3 + pt = q$.

Problem 2.

Consider an $n \times n$ matrix A. A subspace V of $\mathbb{R}^n$ is said to be A-invariant if $A\vec{v}$ is in V for all $\vec{v}$ in V. Describe all the one-dimensional A-invariant subspaces of $\mathbb{R}^n$ in terms of the eigenvectors of A.

Problem 3.

Consider an arbitrary $n \times n$ matrix A. What is the relationship between the characteristic polynomials of A and A^T ? What does your answer tell you about the eigenvalues of A and A^T?

Problem 4.

Suppose matrix A is similar to B. What is the relationship between the characteristic polynomials of A and B? What does your answer tell you about the eigenvalues of A and B?