

**Math 33A**  
**Linear Algebra and Applications**

**Discussion for February 21-25, 2022**

**Problem 1.**

In his groundbreaking text *Ars Magna*, the Italian mathematician Gerolamo Cardano explains how to solve cubic equations. In Chapter XI, he considers the following example:  $x^3 + 6x = 20$ .

- (a) Explain why this equation has exactly one (real) solution. Here, this solution is easy to find by inspection. The point of this exercise is to show a systematic way to find it.
- (b) Cardano explains his method as follows (we are using modern notation for the variables): “I take two cubes  $v^3$  and  $u^3$  whose difference shall be 20, so that the product  $vu$  shall be 2, that is, a third of the coefficient of the unknown  $x$ . Then, I say that  $v - u$  is the value of the unknown  $x$ ”. Show that if  $v$  and  $u$  are chosen as stated by Cardano, then  $x = v - u$  is indeed the solution of the equation  $x^3 + 6x = 20$ .
- (c) Solve the system

$$\begin{aligned}v^3 - u^3 &= 20 \\vu &= 2\end{aligned}$$

to find  $u$  and  $v$ .

- (d) Consider the equation  $x^3 + px = q$ , where  $p$  is positive. Using your previous work as a guide, show that the unique solution of this equation is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Check that this solution can also be written as

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

What can go wrong when  $p$  is negative?

- (e) Consider an arbitrary cubic equation  $x^3 + ax^2 + bx + c = 0$ . Show that the substitution  $x = t - (a/3)$  allows you to write this equation as  $t^3 + pt = q$ .

**Solution:**

- (a) Consider the polynomial  $f(x) = x^3 + 6x - 20$ , so we have  $x^3 + 6x = 20$  if and only if  $f(x) = 0$ . Now  $\lim_{x \rightarrow -\infty} f(x) = -\infty$  and  $\lim_{x \rightarrow +\infty} f(x) = +\infty$ , so  $f$  has at least one root. Since  $f'(x) = 3x^2 + 6$  is always positive,  $f$  is always increasing, so  $f$  has exactly one root.

- (b) Let  $u$  and  $v$  such that  $v^3 - u^3 = 20$  and  $uv = 2$ , and set  $x = v - u$ . Then

$$\begin{aligned}x^3 + 6x &= (v - u)^3 + 6(v - u) = v^3 - 3v^2u + 3vu^2 - u^3 + 6(v - u) \\ &= v^3 - u^3 - 3vu(v - u) + 6(v - u) = 20 - 6(v - u) + 6(v - u) = 20.\end{aligned}$$

(c) From the second equation we have  $u = 2/v$ , and substituting into the first equation gives  $(v^3)^2 - 20v^3 - 8 = 0$ . This has solutions  $v^3 = 10 \pm 6\sqrt{3}$  so  $v = \sqrt[3]{10 \pm 6\sqrt{3}}$ . Now  $u^3 = v^3 - 20 = -10 \pm 6\sqrt{3}$  so  $u = \sqrt[3]{-10 \pm 6\sqrt{3}}$ .

(d) Let  $v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$  and  $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ . Then  $v^3 - u^3 = q$  and  $vu = p/3$ , so since  $x = v - u$  we have

$$\begin{aligned} x^3 + px &= v^3 - 3v^2u + 3vu^2 - u^3 + p(v - u) \\ &= v^3 - u^3 - 3vu(v - u) + p(v - u) = q - p(v - u) + p(v - u) = q. \end{aligned}$$

If  $p$  is negative then  $(q/2) + (p/3)^3$  may be negative, and the equation  $x^3 + px = q$  may have more than one solution.

(e) Let  $x = t - (a/3)$  and assume  $x^3 + ax^2 + bx + c = 0$ , then

$$\begin{aligned} 0 &= x^3 + ax^2 + bx + c = (t - (a/3))^3 + a(t - (a/3))^2 + b(t - (a/3)) + c \\ &= t^3 + (b - (a^2/3))t + (c + (2a^3/27) - (ab/3)) \end{aligned}$$

and thus  $t^3 + (b - (a^2/3))t = (ab/3) - c - (2a^3/27)$ . Setting  $p = b - (a^2/3)$  and  $q = (ab/3) - c - (2a^3/27)$  we have  $t^3 + pt = q$ .

### Problem 2.

Consider an  $n \times n$  matrix  $A$ . A subspace  $V$  of  $\mathbb{R}^n$  is said to be  $A$ -invariant if  $A\vec{v}$  is in  $V$  for all  $\vec{v}$  in  $V$ . Describe all the one-dimensional  $A$ -invariant subspaces of  $\mathbb{R}^n$  in terms of the eigenvectors of  $A$ .

**Solution:** Let  $V$  be a one dimensional  $A$ -invariant subspace, let  $\vec{v} \in V$ . Since  $V$  is one dimensional, we have  $V = \text{span}(\vec{v})$ . Since  $\vec{v} \in V$  then  $A\vec{v} \in V$  so  $A\vec{v} = \lambda\vec{v}$ , meaning that  $\lambda$  is an eigenvalue of  $A$  with associated eigenvector  $\vec{v}$ . Thus every one dimensional  $A$ -invariant subspace  $V$  is the span of an eigenvector of  $A$ .

We now prove the converse, namely, let  $\vec{v}$  be an eigenvector of  $A$  of eigenvalue  $\lambda$ . Then  $V = \text{span}(\vec{v})$  is one dimensional. Moreover  $V$  is an  $A$ -invariant subspace since any  $\vec{w} \in V$  can be written as  $c\vec{v}$  a scalar multiple of  $\vec{v}$ , and thus  $A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v}$  so  $A\vec{w} \in V$  for all  $\vec{w} \in V$ . Thus the span of an eigenvector of  $A$  is a one dimensional  $A$ -invariant subspace.

### Problem 3.

Consider an arbitrary  $n \times n$  matrix  $A$ . What is the relationship between the characteristic polynomials of  $A$  and  $A^T$ ? What does your answer tell you about the eigenvalues of  $A$  and  $A^T$ ?

**Solution:** They coincide. To see this, we use that the determinant of a matrix and the determinant of its transpose coincide, so

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I_n) = \det(A - \lambda I_n)^T \\ &= \det(A^T - \lambda I_n^T) = \det(A^T - \lambda I_n) = f_{A^T}(\lambda). \end{aligned}$$

Thus  $A$  and  $A^T$  have the same eigenvalues, with the same algebraic multiplicities.

**Problem 4.**

Suppose matrix  $A$  is similar to  $B$ . What is the relationship between the characteristic polynomials of  $A$  and  $B$ ? What does your answer tell you about the eigenvalues of  $A$  and  $B$ ?

**Solution:** They coincide. To see this, we use that the determinant is multiplicative and that the determinant of an invertible matrix is the multiplicative inverse of the determinant of its inverse. Suppose that  $B = S^{-1}AS$  for some invertible matrix  $S$ , then

$$\begin{aligned} f_B(\lambda) &= \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n) = \det(S^{-1}AS - \lambda S^{-1}I_n S) \\ &= \det(S^{-1}(A - \lambda I_n)S) = \det(S^{-1}) \det(A - \lambda I_n) \det(S) \\ &= \det(S^{-1}) \det(S) \det(A - \lambda I_n) = \det(A - \lambda I_n) = f_A(\lambda). \end{aligned}$$

Thus  $A$  and  $B$  have the same eigenvalues, with the same algebraic multiplicities.