**Math 33A**
**Linear Algebra and Applications**

**Practice Midterm 2**

**Instructions:** You have 24 hours to complete this exam. There are 7 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. This document is the template where you need to provide your answers. Please print or download this document, complete it in the space provided, show your work in the space provided, clearly box your final answer, and upload a pdf version of this document with your solutions. Do not upload a different document, and do not upload loose paper sheets. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Failure to comply with any of these instructions may have repercussions in your final grade.

Name: ____________________
ID number: ____________________
Section: ____________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 15 | |
| Total: | 100 | |

**Problem 1**. *10pts.*

Determine whether the following statements are true or false.

(a) If matrix $A$ is orthogonal, then matrix $A^2$ must be orthogonal as well.

T

(b) All nonzero symmetric matrices are invertible.

F

(c) If $\vec{u}$ is a unit vector in $\mathbb{R}^n$ and $L = \operatorname{span}(\vec{u})$, then $\operatorname{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

F

(d) If $A$ is an invertible $n \times n$ matrix, then $\det(A^T) = \det(A^{-1})$.

F

(e) There exist real invertible $3 \times 3$ matrices $A$ and $S$ such that $S^{-1}AS = 2A$.

F

**Problem 2**. *15pts.*

Find scalars $a, b, c, d, e, f, g$ such that the vectors

$$\begin{bmatrix} a \\ d \\ f \end{bmatrix}, \quad \begin{bmatrix} b \\ 1 \\ g \end{bmatrix}, \quad \begin{bmatrix} c \\ e \\ 1/2 \end{bmatrix}$$

are orthonormal.

**Solution:** There are multiple solutions, one of them is

$$\begin{bmatrix} 1/2 \\ 0 \\ -\sqrt{3}/2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{bmatrix}.$$

**Problem 3**. *15pts.*

Find the $QR$ factorization of the following matrix.

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

**Solution:** Since the last two columns of the first matrix and the last two rows of the second matrix have no effect on the product, we have

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$

which is the $QR$ factorization of $M$ with

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}.$$

**Problem 4**. *15pts.*
Find all orthogonal matrices of the form

$$\begin{bmatrix} a & b & 0 \\ c & d & 1 \\ e & f & 0 \end{bmatrix}.$$

**Solution:** The first two columns must be orthogonal to the first, so $c = d = 0$. This problem then reduces to find all orthogonal matrices of the form

$$\begin{bmatrix} a & b \\ e & f \end{bmatrix}.$$

The first column is a unit vector, so it can be expressed as

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

where $0 \leq \theta < 2\pi$, having two orthogonal unit vectors

$$\begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

which are the possibilities for the second column. Thus the orthogonal matrices we are looking for are

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \\ \sin(\theta) & -\cos(\theta) & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \\ \sin(\theta) & \cos(\theta) & 0 \end{bmatrix}$$

where $0 \leq \theta < 2\pi$.

**Problem 5**. *15pts.*

Find the least-squares solution $\vec{x}^*$ of the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}.$$

Draw a sketch showing the vector $\vec{b}$, the image of $A$, the vector $A\vec{x}^*$, and the vector $\vec{b} - A\vec{x}^*$.

**Solution:** The least-squares solution is

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

and

$$\vec{b} - A\vec{x}^* = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is perpendicular to $\text{im}(A)$.

**Problem 6**. *15pts.*

Consider a $4 \times 4$ matrix $A$ with rows $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$. If $\det(A) = 8$, find

$$\det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} + 7\vec{v_4} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix}.$$

**Solution:** We know that

$$\det(A) = \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} = 8$$

so since

$$\begin{bmatrix} \vec{v_1} \\ \vec{v_2} + 7\vec{v_4} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix}$$

is found from $A$ by adding a multiple of a row to another row, then

$$\det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} + 7\vec{v_4} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix} = \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} = 8.$$

Alternatively, using the linearity of the determinant, we have

$$\begin{aligned}
\det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} + 7\vec{v_4} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix} &= \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix} + \det \begin{bmatrix} \vec{v_1} \\ 7\vec{v_4} \\ \vec{v_3} - 2\vec{v_4} \\ \vec{v_4} \end{bmatrix} = \\
&= \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} + \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ -2\vec{v_4} \\ \vec{v_4} \end{bmatrix} + \det \begin{bmatrix} \vec{v_1} \\ 7\vec{v_4} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} + \det \begin{bmatrix} \vec{v_1} \\ 7\vec{v_4} \\ -2\vec{v_4} \\ \vec{v_4} \end{bmatrix} = \\
&= \det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} - 2\det \begin{bmatrix} \vec{v_1} \\ \vec{v_2} \\ \vec{v_4} \\ \vec{v_4} \end{bmatrix} + 7\det \begin{bmatrix} \vec{v_1} \\ \vec{v_4} \\ \vec{v_3} \\ \vec{v_4} \end{bmatrix} - 14\det \begin{bmatrix} \vec{v_1} \\ \vec{v_4} \\ \vec{v_4} \\ \vec{v_4} \end{bmatrix} = \\
&= 8 - 2 \cdot 0 + 7 \cdot 0 - 14 \cdot 0 = 8.
\end{aligned}$$

**Problem 7**. *15pts.*

Find the classical adjoint of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 6 & 6 \end{bmatrix}.$$

and use the result to find $A^{-1}$.

**Solution:** We can find

$$\operatorname{adj}(A) = \begin{bmatrix} -6 & 0 & 1 \\ -3 & 5 & -2 \\ 4 & -5 & 1 \end{bmatrix}$$

and $\det(A) = -5$, and thus

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} 6/5 & 0 & -1/5 \\ 3/5 & -1 & 2/5 \\ -4/5 & 1 & -1/5 \end{bmatrix}.$$