DIAGONALIZATION AND THE FIBONACCI SEQUENCE (MATH 33A)

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In this short note, we look at how diagonalization of matrices can be applied to answer some questions about the Fibonacci sequence.

1. The Fibonacci Sequence

The **Fibonacci sequence** is the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

where each term is obtained by adding the two previous terms. For example, we start with $1, 1$. To get the next term, we take $1 + 1 = 2$. For the next term, we take $1 + 2 = 3$. For the next term, we take $2 + 3 = 5$. Here's a picture:

$$\begin{array}{ccccccc} 1 & +1 & =2 & & & & \\ & 1 & +2 & =3 & & & \\ & & 2 & +3 & =5 & & \\ & & & 3 & +5 & =8 & \\ & & & \vdots & & & \end{array}$$

We can define the Fibonacci sequence using a recursive formula: we declare $F_0 = 0$, $F_1 = 1$, and for $n > 1$,

$$\boxed{F_n = F_{n-1} + F_{n-2}.}$$

The Fibonacci sequence comes up every once in a while in pop culture, in part because of its relation to the **Golden Ratio** which has some mystical significance for some people. The purpose of this note is not to convince you of the mystical power of the Fibonacci sequence, nor to detail why this sequence occurs in nature so often. Instead, I want to answer the following question:

Is there a formula for the nth Fibonacci number?

The answer turns out to be yes, and we will see that linear algebra can help us find the answer.

2. Using Linear Algebra

The "trick" explained in this section comes up often in math, especially when solving differential equations. This method can be most directly extended to solve **linear difference equations**, which are equations that relate the difference between consecutive terms in a sequence to the previous terms in the sequence in a linear way.

Instead of looking at a sequence of numbers, we'd like to look at a sequence of *vectors*. We define, for $n \geq 1$,

$$v_n = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} \in \mathbb{R}^n$$

Why would we want to do this? The defining relation for the Fibonacci sequence then gives

$$v_{n+1} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

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If we let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then we have

$$v_{n+1} = Av_n.$$

From this formula, we get that

$$v_n = A^{n-1}v_1.$$

So to find a formula for v_n (which will give a formula for F_n), we can first find a formula for A^n. This is where diagonalization comes in!

3. Diagonalization

We want a formula for A^n. If A was diagonal with entires λ_1 and λ_2, then

$$A^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}.$$

However, A is not diagonal. Let's diagonalize!

The characteristic polynomial of A is

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1.$$

The roots of this polynomial (and hence the eigenvalues of A) are

$$\frac{1 \pm \sqrt{5}}{2}.$$

The number

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

is called the **Golden Ratio**. It satisfies many interesting properties, such as $\phi^2 = \phi + 1$ (squaring ϕ is the same thing as adding 1), or $1/\phi = \phi - 1$ (the reciprocal of ϕ is 1 less). The other root of the characteristic polynomial can be written as $-1/\phi$, or $1 - \phi$.

Now, we find eigenvectors for these eigenvalues, using Gauss–Jordan elimination

$$\begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix} \to \begin{bmatrix} 1 & -\phi \\ 1-\phi & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\phi \\ 0 & 1-\phi(\phi-1) \end{bmatrix} \to \begin{bmatrix} 1 & -\phi \\ 0 & 0 \end{bmatrix},$$

in the last step using that $\phi^2 - \phi - 1 = 0$. So an eigenvector for ϕ is $(\phi, 1)$. Similarly, an eigenvector for $1 - \phi$ is $(1 - \phi, 1)$. So our change of basis matrix is

$$S = \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix}.$$

This has determinant $2\phi - 1 = \sqrt{5} = 5/(2\phi - 1)$. So

$$S^{-1} = \frac{2\phi-1}{5} \begin{bmatrix} 1 & \phi-1 \\ -1 & \phi \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2\phi-1 & -\phi+3 \\ 1-2\phi & \phi+2 \end{bmatrix}.$$

Now here comes the magic. We know that

$$A = S \begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix} S^{-1}.$$

But then

$$A^2 = S \begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix} S^{-1} S \begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix} S^{-1} = S \begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix}^2 S^{-1}.$$

In general,

$$A^n = S \begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix}^n S^{-1} = S \begin{bmatrix} \phi^n & 0 \\ 0 & (1-\phi)^n \end{bmatrix} S^{-1}.$$

So

$$
\begin{aligned}
v_n &= A^{n-1} v_1 \\
&= S \begin{bmatrix} \phi^{n-1} & 0 \\ 0 & (1-\phi)^{n-1} \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^{n-1} & 0 \\ 0 & (1-\phi)^{n-1} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 2\phi-1 & -\phi+3 \\ 1-2\phi & \phi+2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^{n-1} & 0 \\ 0 & (1-\phi)^{n-1} \end{bmatrix} \begin{bmatrix} 2\phi-1 \\ 1-2\phi \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2\phi^n - \phi^{n-1} \\ (1-\phi)^n + (1-\phi)^{n-2} \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 2\phi^{n+1} - \phi^n + (1-\phi)^{n+1} + (1-\phi)^{n-1} \\ 2\phi^n - \phi^{n-1} + (1-\phi)^n + (1-\phi)^{n-2} \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} (2\phi-1)\phi^n + (1-2\phi)(1-\phi)^n \\ (2\phi-1)\phi^{n-1} + (1-2\phi)(1-\phi)^{n-1} \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^n - (1-\phi)^n \\ \phi^{n-1} - (1-\phi)^{n-1} \end{bmatrix}.
\end{aligned}
$$

Remembering that $v_n = (F_n, F_{n-1})$, we have

$$\boxed{F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}.}$$

4. Consequences

We note that $|1-\phi| \approx -0.618$, so for $n \geq 1$,

$$\left| \frac{(1-\phi)^n}{\sqrt{5}} \right| < \frac{1}{2}.$$

So we can conclude that

$$\left| F_n - \frac{\phi^n}{\sqrt{5}} \right| < \frac{1}{2}.$$

In other words, F_n is the closest integer to $\phi^n/\sqrt{5}$. This makes some calculations much faster. For instance, $\phi^2 0/\sqrt{5} \approx 6765.00000$, so $F_{20} = 6765$. In fact, we note that

$$
\begin{aligned}
\left| \frac{F_{n+1}}{F_n} - \phi \right| &= \left| \frac{\phi^{n+1} - (1-\phi)^{n+1} - \phi^{n+1} + \phi(1-\phi)^n}{\phi^n - (1-\phi)^n} \right| \\
&= \left| \frac{(1-\phi)^{n+1} + (1-\phi)^{n-1}}{\phi^n - (1-\phi)^n} \right| \\
&\leq |1-\phi|^n.
\end{aligned}
$$

So we can approximate ϕ pretty closely by the ratio of consecutive Fibonacci numbers. For example,

$$F_{21}/F_{20} = 10946/6765 \approx 1.61803 \approx \phi.$$

We can say something even stronger: for $n \geq 4$,

$$|F_{n+1} - \phi F_n| = |1-\phi|^n |1-2\phi| = \sqrt{5}|1-\phi|^n < \frac{1}{2}.$$

So given F_n for any $n \geq 4$, we can find F_{n+1} just by multiplying by ϕ and rounding. We have to be careful in practice, though, because we only calculate ϕ up to some finite precision, and the rounding that a calculator does might give a slightly different answer.

By slightly changing our recurrence relation for F_n, we can find other sequences that approximate roots to other polynomials. For example, suppose we wanted to find a root to

$$x^3 - x - 1.$$

Using Cardano's method, we could find that

$$\sqrt[3]{\frac{1+\sqrt{\frac{23}{27}}}{2}} + \sqrt[3]{\frac{1-\sqrt{\frac{23}{27}}}{2}}$$

is a root. However, if you didn't have a calculator, this might be hard to estimate. Instead, we consider the sequence defined by

$$a_{n+3} = a_{n+1} + a_n.$$

We pick some small random starting numbers, like $1, 1, 1$, and calculate some terms

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \ldots$$

Taking the ratio of two consecutive large terms approximates a root to the polynomial. Indeed,

$$(86/65)^3 - (86/65) - 1 \approx -0.007.$$

So $86/65$ approximates a root to the polynomial $x^3 - x - 1$.

The formula for F_n also gives us some interesting number theoretic facts. For example, some fancier number theory from the formula shows that if n divides m, then F_n divides F_m. For example, we have $F_{20} = 6765$ is divisible by $F_5 = 5$. The reverse question turns out to be a lot harder. For example, it follows from above that if F_n is a prime number, then n is a prime number. But the converse is not true, and it is unknown if there are even infinitely many Fibonacci numbers that are primes.